1. (10 points) The textbook (section 6.5) asserts without proof that the wavelength $\lambda_m$ at which a reflector telescope has maximum gain is given by $\lambda_m = 4\pi\sigma$, where $\sigma$ is the rms surface error. Derive this equation.

**Solution:**

$$G = \frac{4\pi A}{\lambda^2}$$

But with an imperfect surface, $A = \eta_s A_e$. In class, we derived

$$\eta_s = \exp \left[ -\left( \frac{4\pi\sigma}{\lambda} \right)^2 \right],$$

so therefore,

$$G = \frac{4\pi A_e}{\lambda^2} \exp \left[ -\left( \frac{4\pi\sigma}{\lambda} \right)^2 \right].$$

To find the maximum gain, we simply differentiate $G$ with respect to $\lambda$ and set it equal to 0, solving for $\lambda$.

$$\frac{dG}{d\lambda} = 0 = \frac{2(4\pi)^3 A_e \sigma^2 \exp \left[ -\left( \frac{4\pi\sigma}{\lambda} \right)^2 \right]}{\lambda^5} - \frac{8\pi A_e \exp \left[ -\left( \frac{4\pi\sigma}{\lambda} \right)^2 \right]}{\lambda^3}$$

$$= -\frac{8\pi A_e \exp \left[ -\left( \frac{4\pi\sigma}{\lambda} \right)^2 \right]}{\lambda^5} \left( \lambda^2 - (4\pi\sigma)^2 \right)$$

$$= \lambda^2 - (4\pi\sigma)^2$$

and therefore,

$$\lambda_m = 4\pi\sigma$$

for the maximum antenna gain.

2. (5 points) In class, we showed that a 1-D aperture with constant illumination produces a power pattern described by:

$$P(\theta) \propto \text{sinc}^2 \left( \frac{\theta D}{\lambda} \right).$$

With $D = 10$ m and $\lambda = 10$ cm, use numerical techniques (i.e. using Matlab, IDL, Mathematica, Maple, Python+Scipy, Perl, Octave, Yorick, etc.), compute $\theta$ and $P(\theta)$ to at least 4 significant figures at the peaks of the first 2 sidelobes. Express them in dB as well as in relative terms (as compared to the main beam). Verify your results analytically.

**Solution:** For the “analytical” portion, using $x = \pi\theta D/\lambda$, a local maximum will occur at:

$$\frac{dP(\theta)}{d\theta} = 0 = \frac{d}{d\theta} \left[ \frac{(\sin(x))^2}{x^2} \right]$$

$$= \frac{2\sin(x)\cos(x)}{x^2} - \frac{2(sin(x))^2}{x^3}$$

That simplifies to $\cos(x) = \sin(x)/x$, or $x = \tan(x)$, which can easily be solved using any reasonable root-finding algorithm to give: $x_1 = 4.49341$ and $x_2 = 7.72525$ for the first two maxima (i.e. sidelobes). For our aperture and wavelength, we get: $\theta_1 = 0.014303$ and $\theta_2 = 0.024590$ radians, which corresponds to $P(\theta_1) = 0.047190$ (-13.261 dB) and $P(\theta_2) = 0.016480$ (-17.830 dB).
3. (10 points) For a circular aperture with uniform illumination, the normalized power pattern $P_n(\theta)$ is known as the *Airy disk* and is described in the textbook by eqn. 6.27 (remember that $u = \sin \theta$ where $\theta$ is the angle between the optical axis and the direction in question).

(a) (4 points) As in problem 2, numerically compute $\theta$ and $P_n(\theta)$ to at least 4 significant figures for the first 2 sidelobes with $D = 100 \text{ m}$ and $\lambda = 5 \text{ cm}$.

**Solution:** This is a relatively simple root-finding problem once you look up how to take derivatives of Bessel functions. Using $x = uD/\lambda$, where $u = \sin(\theta)$, we have:

\[
\frac{d}{dx} \sqrt{P(x)} = \frac{d}{dx} \left[ \frac{2J_1(\pi x)}{\pi x} \right] = -\frac{2J_2(\pi x)}{x} = 0.
\]

Note that we don’t need to take the derivative of the square of the function, as the derivatives will be zero at the square-root of the $P(x)$ as well. The result can be handled just like in problem 3 (assuming that you can find an accurate Bessel function routine).

The correct answers are:

- $\theta_1 = 8.1736 \times 10^{-3} \text{ rad}$, $P(\theta_1) = 0.017498$ (−17.570 dB)
- $\theta_2 = 1.3396 \times 10^{-2} \text{ rad}$, $P(\theta_2) = 0.0041580$ (−23.811 dB)

The attached figure shows the Airy function.

(b) (3 points) Numerically determine the aperture’s beam solid angle $\Omega_A$.

**Solution:** This problem demands that you (carefully!) integrate the function

\[
\Omega_A = \int_0^{2\pi} \int_0^{\pi/2} P_n(\theta, \phi) \sin(\theta) d\theta d\phi = 2\pi \int_0^{\pi/2} P_n(\theta, \phi) \sin(\theta) d\theta
\]

where we have integrated over only the top half-plane (for a realistic reflecting antenna). The tricky part is that most of the area of the integrand is near $\theta = 0$, but there is still significant power at large angles. This means that you need a stable integration scheme (i.e. the trapezoidal rule using *tiny* steps will likely not cut it as you will run into floating-point rounding errors and loss of precision).

The correct answer is: $\Omega_A = 3.1831 \times 10^{-7} \text{ sr}$.

(c) (3 points) Compute the main beam solid angle $\Omega_{MB}$ and the beam efficiency $\eta_B$ assuming that the main beam is defined as everything within the first null of eqn. 6.27.

**Solution:** Now we need to integrate

\[
\Omega_{MB} = 2\pi \int_{\theta_{\text{main lobe}}}^{\theta_{\text{main lobe}}} P_n(\theta, \phi) d\Theta,
\]

after determining $\theta_{\text{main lobe}}$ using a root-finding algorithm ($\theta_{\text{main lobe}} = 6.09835 \times 10^{-4} \text{ rad}$). The correct answer is: $\Omega_{MB} = 2.6668 \times 10^{-7} \text{ sr}$.

And therefore, $\eta_B = \Omega_{MB}/\Omega_A = 0.8378$. 

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