ASTR 534: Radio Astronomy Problem Set 3 Due: 25 September, 2006

1. (10 points) The textbook (section 6.5) asserts without proof that the wavelength $\lambda_{\rm m}$ at which a reflector telescope has maximum gain is given by $\lambda_{\rm m} = 4\pi\sigma$, where σ is the rms surface error. Derive this equation.

Solution:

$$G = \frac{4\pi A}{\lambda^2}$$

But with an imperfect surface, $A = \eta_{\rm s} A_{\rm e}$. In class, we derived

$$\eta_{\rm s} = \exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right],$$

so therefore,

$$G = \frac{4\pi A_{\rm e}}{\lambda^2}\exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right].$$

To find the maximum gain, we simply differentiate G with respect to λ and set it equal to 0, solving for λ.

$$\begin{aligned}\frac{dG}{d\lambda} = 0 &= \frac{2(4\pi)^3 A_{\rm e}\sigma^2 \exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right]}{\lambda^5} - \frac{8\pi A_{\rm e}\exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right]}{\lambda^3} \\ &= -\frac{8\pi A_{\rm e}\exp\left[-\left(\frac{4\pi\sigma}{\lambda}\right)^2\right]\left(\lambda^2 - (4\pi\sigma)^2\right)}{\lambda^5} \\ &= \lambda^2 - (4\pi\sigma)^2\end{aligned}$$

and therefore,

$$\lambda_{\rm m} = 4\pi\sigma$$

for the maximum antenna gain.

2. (5 points) In class, we showed that a 1-D aperture with constant illumination produces a power pattern described by:

$$P(\theta) \propto \mathrm{sinc}^2\left(\frac{\theta D}{\lambda}\right).$$

With $D = 10\,\mathrm{m}$ and $\lambda = 10\,\mathrm{cm}$, use *numerical techniques* (i.e. using Matlab, IDL, Mathematica, Maple, Python+Scipy, Perl, Octave, Yorick, etc.), compute θ and $P(\theta)$ to at least 4 significant figures at the peaks of the first 2 sidelobes. Express them in dB as well as in relative terms (as compared to the main beam). Verify your results analytically.

Solution: For the "analytical" portion, using $x = \pi\theta D/\lambda$, a local maximum will occur at:

$$\begin{aligned}\frac{dP(\theta)}{d\theta} = 0 &= \frac{d}{d\theta}\left[\left(\frac{\sin(x)}{x}\right)^2\right] \\ &= \frac{2\sin(x)\cos(x)}{x^2} - \frac{2(sin(x))^2}{x^3}\end{aligned}$$

That simplifies to $\cos(x) = \sin(x)/x$, or $x = \tan(x)$, which can easily be solved using any reasonable root-finding algorithm to give: $x_1 = 4.49341$ and $x_2 = 7.72525$ for the first two maxima (i.e side-lobes). For our aperture and wavelength, we get: $\theta_1 = 0.014303$ and $\theta_2 = 0.024590$ radians, which corresponds to $P(\theta_1) = 0.047190$ (-13.261 dB) and $P(\theta_2) = 0.016480$ (-17.830 dB).

3. (10 points) For a circular aperture with uniform illumination, the normalized power pattern $P_n(\theta)$ is known as the *Airy disk* and is described in the textbook by eqn. 6.27 (remember that $u = \sin\theta$ where θ is the angle between the optical axis and the direction in question).

   (a) (4 points) As in problem 2, numerically compute θ and $P_n(\theta)$ to at least 4 significant figures for the first 2 sidelobes with $D = 100\,\mathrm{m}$ and $\lambda = 5\,\mathrm{cm}$.

   Solution: This is a relatively simple root-finding problem once you look up how to take derivatives of Bessel functions. Using $x = uD/\lambda$, where $u = \sin(\theta)$, we have:

$$\frac{d}{dx}\sqrt{P(x)} = \frac{d}{dx}\left[\frac{2J_1(\pi x)}{\pi x}\right] = -2\frac{J_2(\pi x)}{x} = 0.$$

   Note that we don't need to take the derivative of the square of the function, as the derivatives will be zero at the square-root of the $P(x)$ as well. The result can be handled just like in problem 3 (assuming that you can find an accurate Bessel function routine).

   The correct answers are:

$$\theta_1 = 8.1736 \times 10^{-3}\,\mathrm{rad} \qquad P(\theta_1) = 0.017498 \quad (-17.570\,\mathrm{dB})$$

$$\theta_2 = 1.3396 \times 10^{-2}\,\mathrm{rad} \qquad P(\theta_2) = 0.0041580 \quad (-23.811\,\mathrm{dB})$$

   The attached figure shows the Airy function.

   (b) (3 points) Numerically determine the aperture's beam solid angle Ω_A.

   Solution: This problem demands that you (carefully!) integrate the function

$$\Omega_\mathrm{A} = \int_0^{2\pi}\int_0^{\pi/2} P_n(\theta,\phi)\sin(\theta)d\theta d\phi = 2\pi\int_0^{\pi/2} P_n(\theta,\phi)\sin(\theta)d\theta$$

   where we have integrated over only the top half-plane (for a realistic reflecting antenna).

   The tricky part is that most of the area of the integrand is near $\theta = 0$, but there is still significant power at large angles. This means that you need a stable integration scheme (i.e. the trapezoidal rule using *tiny* steps will likely not cut it as you will run into floating-point rounding errors and loss of precision).

   The correct anwser is: $\Omega_\mathrm{A} = 3.1831 \times 10^{-7}\,\mathrm{sr}$.

   (c) (3 points) Compute the main beam solid angle Ω_MB and the beam efficiency η_B assuming that the main beam is defined as everything within the first null of eqn. 6.27.

   Solution: Now we need to integrate

$$\Omega_\mathrm{MB} = 2\pi\int_0^{\theta_\text{main lobe}} P_n(\theta,\phi)d\Theta,$$

   after determining $\theta_\text{main lobe}$ using a root-finding algorithm ($\theta_\text{main lobe} = 6.09835 \times 10^{-4}\,\mathrm{rad}$).

   The correct anwser is: $\Omega_\mathrm{MB} = 2.6668 \times 10^{-7}\,\mathrm{sr}$.

   And therefore, $\eta_\mathrm{B} = \Omega_\mathrm{MB}/\Omega_\mathrm{A} = 0.8378$.