

1. You would like to measure the temperature of Mars with an rms uncertainty of about 1% by observing it at  $\nu_{\text{RF}} \approx 10$  GHz. The angular diameter of Mars is  $\theta_M = 18''$ , and its 10 GHz flux density is 4.2 Jy. You have a radio telescope whose paraboloidal mirror has diameter  $D = 25$  m and aperture efficiency  $\eta_A = 0.70$ . The single-channel total-power receiver is connected to a feed sensitive to right circular polarization, and its RF bandwidth is 100 MHz. The receiver noise temperature is  $T_{\text{RX}} = 18$  K, the atmosphere adds about 3.5 K, the microwave background 3 K, and spillover pickup of ground radiation is about 11 K.

- (a) Show that Mars is a “point source” for your observation; that is,  $\theta_M \ll \theta_{\text{FWHM}}$ , the telescope beamwidth between half-power points.

**Solution:** Since  $\theta_{\text{FWHM}} \approx \lambda/D$ , and  $\lambda = c/\nu$ , we have

$$\begin{aligned} \theta_{\text{FWHM}} &\approx \frac{c}{\nu D} \\ &= \frac{3 \times 10^8}{10^{10} \times 25} \\ &= 0.12 \text{ radians} \\ &= 6.9^\circ \gg 18'' = \theta_M \end{aligned}$$

Note that for radio telescopes,  $\theta_{\text{FWHM}} = 1.2\lambda/D$ , but here an order-of-magnitude calculation works as well.

- (b) What antenna temperature contribution  $T_A$  do you expect from Mars?

**Solution:**

$$\begin{aligned} A_e &= \eta_A \frac{\pi D^2}{4}, \text{ and} \\ \frac{1}{2} A_e S_\nu &= k T_A \\ \Rightarrow T_A &= \eta_A \frac{\pi D^2}{8k} S_\nu \\ &= 0.70 \times \frac{3.14 \times 25^2}{8 \times 1.38 \times 10^{-23}} \times 4.2 \times 10^{-26} \\ &= 0.52 \text{ K.} \end{aligned}$$

- (c) What is the system noise temperature  $T_{\text{sys}}$  when the telescope is pointing at Mars?

**Solution:**

$$\begin{aligned} T_{\text{sys}} &= T_{\text{cmb}} + T_A + T_{\text{atm}} + T_{\text{spillover}} + T_{\text{RX}} \\ &= 3 + 0.52 + 3.5 + 11 + 18 \\ &= 36.02 \text{ K} \end{aligned}$$

- (d) If the receiver has perfect gain stability, how long must you point the telescope at Mars?

**Solution:** Using the radiometer equation,

$$\begin{aligned}\frac{\sigma_T}{T_{\text{sys}}} &\approx \frac{1}{\sqrt{\Delta\nu_{\text{RF}}\tau}} \\ \frac{0.01T_M}{36.02} &= \frac{1}{\sqrt{10^8 \times \tau}} \\ \Rightarrow \tau &= \frac{36.02^2}{10^{-4} \times 0.52^2 \times 10^8} \\ &= 0.48 \text{ s}\end{aligned}$$

- (e) Estimate the maximum rms receiver gain fluctuation  $\Delta G/G$  that you can tolerate during this observation.

**Solution:**

$$\begin{aligned}\frac{\Delta G}{G} &= \frac{\sigma_T}{T_{\text{sys}}} \\ &= 1.4 \times 10^{-4}\end{aligned}$$

- (f) If the receiver gain stability is not good enough for a total-power observation and you are forced to Dicke switch, how long must you point the telescope at Mars?

**Solution:** The Dicke-switch radiometer equation is

$$\sigma_T \approx 2T_{\text{sys}} \frac{1}{\sqrt{\Delta\nu_{\text{RF}}\tau}}$$

Using this equation and the radiometer equation for total-power observations (see above), we get

$$\begin{aligned}\frac{\tau_{\text{Dicke-switch}}}{\tau_{\text{total-power}}} &= 4 \\ \Rightarrow \tau &= 1.92 \text{ s}\end{aligned}$$

2. For pulsar timing observations, astronomers “fold” the data modulo the pulse period. If the number of bins in the pulse profile is such that the pulse fits perfectly into a single bin, then for simple single-peaked pulse profiles, this technique is practically an optimal “matched-filter” for the signal.

- (a) If we assume a rectangular (i.e. top-hat) pulse shape of width  $W$  seconds, measured peak amplitude  $T_{\text{peak}}$  Kelvin, and spin period  $P$  seconds, derive the signal-to-noise ratio ( $S/N = T_{\text{peak}}/T$ ) as a function of the integration time  $\tau$ , the observed bandwidth  $\Delta\nu_{\text{RF}}$ ,  $W$  and  $P$ . You can assume (as is nearly always true) that  $T_{\text{peak}} \ll T_{\text{sys}}$ .

**Solution:** The total number of pulses observed in time  $\tau$ ,  $N = \tau/P$ . For simplicity, we can assume that  $N$  as defined above is an integer. Then, the observation of pulses with peak  $T_{\text{peak}}$  for a time  $\tau$  is equivalent to observing a source with peak  $\sqrt{N}T_{\text{peak}}$  for a time  $P$ .

Now, during the time of the pulse, the rms noise is given by

$$\sigma_{\text{pulse}} = \frac{T_{\text{sys, pulse}}}{\sqrt{\Delta\nu_{\text{RF}} W}},$$

and during the time of no pulse, the rms noise is given by

$$\sigma_{\text{baseline}} = \frac{T_{\text{sys, baseline}}}{\sqrt{\Delta\nu_{\text{RF}}(P - W)}},$$

where  $T_{\text{sys, pulse}} - T_{\text{sys, baseline}} = \sqrt{N}T_{\text{peak}}$ . Since  $T_{\text{peak}} \ll T_{\text{sys}}$ , we can set the numerators of both the equations equal to  $T_{\text{sys}}$ .

Then, the total noise power is given by:

$$\begin{aligned} \sigma_{\text{tot}} &= \sqrt{\sigma_{\text{pulse}}^2 + \sigma_{\text{baseline}}^2} \\ &= T_{\text{sys}} \sqrt{\frac{1}{\Delta\nu_{\text{RF}} W} + \frac{1}{\Delta\nu_{\text{RF}}(P - W)}} \\ &= \frac{T_{\text{sys}}}{\sqrt{\Delta\nu_{\text{RF}}}} \sqrt{\frac{P}{W(P - W)}}, \end{aligned}$$

and  $S/N$  is given by

$$\begin{aligned} S/N &= \frac{\sqrt{N}T_{\text{peak}}}{\sigma_{\text{tot}}} \\ &= \frac{T_{\text{peak}}\sqrt{\Delta\nu_{\text{RF}}\tau}}{T_{\text{sys}}} \frac{\sqrt{W(P - W)}}{P} \end{aligned}$$

- (b) For  $W \ll P$ , how does  $S/N$  scale with the pulsar “duty-cycle”,  $W/P$ ?

**Solution:** For  $W \ll P$ ,  $(P - W)/P \approx 1$ , and we get

$$S/N \propto (W/P)^{-\frac{1}{2}}$$

- (c) For a few points of extra credit, explain how this applies to pulsar searches, which are conducted in the frequency domain after a Fourier transform (and where we don’t know  $W$  and  $P$  a priori).

**Solution:** The Fourier transform of a series of periodic pulses consists of peaks at the frequency corresponding to the period, and at that frequency’s harmonics. We can look at the Fourier transform of the time signal, and search for harmonics for pulsar searches.