1. (2 points) The Crab pulsar has a very steep radio spectral index of approximately -3 (i.e. ν^{-3}) over a frequency range from 10 MHz to 10 GHz. If the distance to the Crab pulsar is $\sim$2 kpc, the measured flux density at 400 MHz is 650 mJy, and the spin-down luminosity (i.e. $\dot{E}$) as derived in class is 4×10^{38} erg s^{-1}, what fraction of $\dot{E}$ does the radio emission account for?

Solution: The distance to the Crab nebula is $D = 2\,\mathrm{kpc} = 6.2\times10^{21}$ cm.

$$\begin{aligned} L_{\text{radio}} &= 4\pi D^2 \int_{\text{radio}} S_{\text{radio}}\, d\nu \\ &= 4\pi \left(6.2\times10^{21}\right)^2 \int_{10^7}^{10^{10}} 0.65 \left(\frac{\nu}{4\times10^8}\right)^{-3} d\nu \\ &= 2\times10^{47} \left[\frac{\nu^{-2}}{-2}\right]_{10^7}^{10^{10}} \\ &= 1\times10^{33}\,\mathrm{ergs\,s^{-1}} \end{aligned}$$

Therefore the fraction is $\frac{L_{\text{radio}}}{\dot{E}} = \frac{1\times10^{33}}{4\times10^{38}} = 2.5\times10^{-6}$ — very small!

2. (2 points) Pulsar astronomers parameterize pulsar spin-down in a model-independent way with the relation $\dot{\Omega} \propto \Omega^n$, where n is known as the braking index. Derive a functional form for n in terms of the three observables: Ω, $\dot{\Omega}$, and $\ddot{\Omega}$. What value is n for magnetic dipole radiation?

Solution:

$$\ddot{\Omega} = n\Omega^{n-1}\dot{\Omega},$$

but from the question, $\Omega^{n-1} = \dot{\Omega}/\Omega$, and so

$$\ddot{\Omega} = n\frac{\dot{\Omega}}{\Omega}\dot{\Omega},$$

and therefore

$$n = \frac{\ddot{\Omega}\Omega}{\dot{\Omega}^2}.$$

For magnetic dipole radiation,

$$\dot{E} = I\Omega\dot{\Omega} = \frac{2}{3c^3}(BR^3\sin\alpha)^2\Omega^4,$$

therefore, $\dot{\Omega} \propto \Omega^3$, and therefore $n = 3$.

3. (5 points) The spin-down relation in terms of period P is $\dot{P} \propto P^{2-n}$.

   (a) (2 points) Use this relation (assuming an initial spin period P_0) to show that the pulsar age T is

$$T = \frac{P}{(n-1)\dot{P}}\left[1 - \left(\frac{P_0}{P}\right)^{n-1}\right]$$

   without assuming magnetic dipole braking, constant magnetic field strength, or $P_0 \ll P$.

Solution:

$$\frac{dP}{dt} = P^{2-n}$$

$$\int_{P_0}^{P} P^{-(2-n)}\, dP = \int_0^T dt$$

$$\left.\frac{P^{n-1}}{n-1}\right|_{P_0}^{P} = T$$

$$\frac{P^{n-1}}{n-1} - \frac{P_0^{n-1}}{n-1} = T$$

$$\frac{P^{n-1}}{n-1}\left[1 - \left(\frac{P_0}{P}\right)^{n-1}\right] = T,$$

but since $\dot{P} = P^{2-n}$, therefore, $\dot{P}^{-1} = P^{n-2}$ and $P\dot{P}^{-1} = P^{n-1}$, which gives us:

$$T = \frac{P}{(n-1)\dot{P}}\left[1 - \left(\frac{P_0}{P}\right)^{n-1}\right]$$

(b) (1 point) Show that for the common assumptions of $n = 3$ and $P_0 \ll P$ this reduces to the characteristic age, τ_c, as derived in class.

Solution: If $P_0 \ll P$ and $n = 3$, then $\left(\frac{P_0}{P}\right)^{n-1} \sim 0$ giving $T = \tau_c = \frac{P}{2\dot{P}}$.

(c) (2 points) If a 100 ms pulsar is found with $\tau_c = 30$ kyr in a supernova remnant where historical or kinematic data suggest a true age of only ~2 kyr, what does this imply about the pulsar? Can this discovery constrain the braking index? Why or why not?

Solution: Given the definition for τ_c, we know that $\frac{P}{\dot{P}} = 2\tau_c = 60$ kyr. We can therefore substitute 60 kyr for $\frac{P}{\dot{P}}$ into the equation for T:

$$T = 2\,\text{kyr} = \frac{60\,\text{kyr}}{(n-1)}\left[1 - \left(\frac{P_0}{P}\right)^{n-1}\right].$$

Rearranging and solving for $\frac{P_0}{P}$, we get:

$$\frac{P_0}{P} = \left[1 - \frac{(n-1)}{30}\right]^{\frac{1}{n-1}}.$$

For any reasonable value for the braking index n, $|n-1|/30 \ll 1$, and we can expand to get:

$$\frac{P_0}{P} \approx 1 - \frac{(n-1)}{(n-1)30} = 1 - \frac{1}{30} = 0.967.$$

This means that regardless of the value of the braking index, the initial spin period P_0 was very close to the current value of the spin period P. Therefore, these pulsar measurements do not constrain the braking index at all. And in addition, the pulsar shows that the commonly used assumption of $P_0 \ll P$ is not always valid (in fact, there are several real pulsars just like this one).

4. (6 points) Pulsar dispersion and scattering

(a) (1 point) Using the relation that we derived in class for the dispersive delay t, derive a simple relation for the total amount of smearing Δt in time that will occur over a small frequency bandwidth BW due to uncorrected dispersion. You may assume that the slope of the dispersion curve (i.e. dt/df) is constant over the bandwidth in question.

Solution: From class, the dispersive delay t is:

$$t = 4.149 \times 10^3 \,\mathrm{DM}\, \nu^{-2}$$

where t is in seconds and ν is in MHz. In differential form, this relation is:

$$dt = -8.3 \times 10^3 \,\mathrm{DM}\, \nu^{-3}\, d\nu.$$

However, if the bandwidth in question is small, then BW$\approx d\nu$ and the smearing Δt is:

$$\Delta t = 8.3 \times 10^3 \,\mathrm{DM}\,\mathrm{BW}\, \nu^{-3}.$$

(Note that smearing is always positive, so we just get rid of the negative sign).

(b) (1 point) Estimate the value for the smearing at the central frequency of 1380 MHz for the 3 MHz channels in Figure 6 of the class notes if DM=100 pc cm^{-3} and DM=1000 pc cm^{-3}.

Solution: Using the above relation:

$$\Delta t = 8.3 \times 10^3\,(100\,\mathrm{pc\,cm^{-3}})\,(3\,\mathrm{MHz})\,(1380\,\mathrm{MHz})^{-3} \approx 950\,\mu\mathrm{s}$$

and

$$\Delta t = 8.3 \times 10^3\,(1000\,\mathrm{pc\,cm^{-3}})\,(3\,\mathrm{MHz})\,(1380\,\mathrm{MHz})^{-3} \approx 9.5\,\mathrm{ms}$$

(c) (1 point) Above approximately what DM will the above system have too much smearing to detect a 2 ms pulsar?

Solution: In order to detect a periodic signal, the Nyquist theorem says that we must have at least 2 samples per period. For a 2 ms pulsar, this means that the smearing can be no more than 1 ms. With the above relation we get:

$$0.001\,\mathrm{s} = 8.3 \times 10^3\,\mathrm{DM}\,(3\,\mathrm{MHz})\,(1380\,\mathrm{MHz})^{-3}$$

Solving for DM gives us: DM $\approx 105\,\mathrm{pc\,cm^{-3}}$.

(d) (1 point) Estimate the DM of the pulsar in Figure 6 (which uses the above system) if its spin period is 2 ms, 400 ms, and 8 s.

Solution: From the top of the band to the bottom, the pulses are delayed by approximately 2.15 full periods. The highest and lowest frequency channels are centered at 1380 MHz $\pm \left(\frac{96}{2} - 0.5\right) \times 3$ MHz, or 1522.5 MHz and 1237.5 MHz.
Using the delay equation from class we get that the differential delay Δt between two frequencies ν_1 and ν_2 is:

$$\Delta T = t_2 - t_1 = 4.149 \times 10^3\,\mathrm{DM}\,\left(\nu_2^{-2} - \nu_1^{-2}\right).$$

Solving for DM gives us:

$$\mathrm{DM} = \frac{\Delta T}{4.149 \times 10^3}\left(\nu_2^{-2} - \nu_1^{-2}\right)^{-1},$$

or in our case,

$$\mathrm{DM} = \frac{2.15\,P}{4.149 \times 10^3} \left(1237.5^{-2} - 1522.5^{-2}\right)^{-1}.$$

Substituting, we get:

$$P = 2\,\mathrm{ms} : \mathrm{DM} = 4.68\ \mathrm{pc\,cm^{-3}}$$
$$P = 400\,\mathrm{ms} : \mathrm{DM} = 935\ \mathrm{pc\,cm^{-3}}$$
$$P = 8\,\mathrm{s} : \mathrm{DM} = 1.87 \times 10^4\ \mathrm{pc\,cm^{-3}}$$

(e) (2 points) Use Figure 9 in the class notes to estimate how pulse scatter-broadening scales as a function of observing frequency ν. Do this using least-squares fitting of measurements from at least 4 or 5 of the profiles. Show a plot of the measurements and the fit and the code that you used to do the fitting.

Solution: Using an image viewer, we can measure the FWHM of the different profiles and estimate the errors.

Frequency (MHz)	FWHM (pixels)	Broadening (pixels)
1408 MHz	12 ± 1	N/A (see below)
610 MHz	15 ± 1	3 ± 1
408 MHz	40 ± 2	28 ± 2
325 MHz	84 ± 3	72 ± 3
243 MHz	250 ± 8	238 ± 8

Since we know that there must be some intrinsic width to the profile, we don't want to include that in the fit. At 1408 MHz, the pulse seems resolved and the two components are very nearly Gaussian. Therefore, the broadening is very small at that frequency and we can assume that the intrinsic width is the FWHM at 1408 MHz. In the table above, I've simply subtracted that width from the FWHM measurements at each of the other frequencies. These are the values that I then fit to the relation:

$$\Delta T = C\nu^{\alpha}.$$

The results of the fit show that $\alpha \sim -4.2$ (see figure).

5. (8 points) The vast majority of pulsars have been found using Fourier analysis. This problem will make you brush up on your FFT knowledge. For an observation with N samples of duration dt (making a total integration time $T = N\ dt$), and a pulsar of spin period $P_{\rm PSR} = 1/f_{\rm PSR}$, where $f_{\rm PSR}$ is the spin frequency:

(a) (1 point) How many independent Fourier bins will there be in an FFT of the real-valued time series?

> **Solution:** If the input data were complex numbers, there would be N independent Fourier bins (each of which is complex) on output (i.e. no information is created or destroyed by an FFT). However, for *real* input data, there are actually only $N/2+1$ *independent* Fourier bins (the "negative" frequencies are complex conjugates of the positive frequencies and are therefore not independent).
>
> So where does that extra independent frequency come from if we are not creating information? It turns out that there are two Fourier amplitudes (the zeroth frequency and the Nyquist frequency) whose complex parts are always zero. That means that the total number of independent *values* (i.e. real and complex parts) is N, the same as in the input values, and not $N+2$.

(b) (1 point) What is the Nyquist frequency? What bin number in the FFT does it correspond to?

> **Solution:** The Nyquist frequency is 1/2 the sampling frequency. It is the highest frequency in the FFT and is in bin number N/2.

(c) (1 point) What is the significance of the zeroth frequency bin?

> **Solution:** It is the "DC"-component of the input signal (i.e. if you add up all the input values, that is the value of the zeroth frequency bin). Note that depending on how the FFT gets normalized, this component might be the *average* input value instead.

(d) (1 point) What is the frequency spacing of the Fourier bins in Hz?

> **Solution:** Since the frequencies in a Fourier bin from a FFT correspond to integer numbers of sinusoid cycles over the input time T (i.e. bin 10 corresponds to exactly 10 complete sinusoid oscillations in the input data), the frequency spacing must be $1/T$. (note that it is *not* $1/dt$!)

(e) (1 point) In what Fourier bin will the pulsar's fundamental (i.e. 1st) harmonic show up? The 2nd harmonic?

> **Solution:** The pulsar's fundamental and 2nd harmonics must be at $f_{\rm PSR}$ Hz and $2f_{\rm PSR}$ Hz respectively. To convert these to bins, we simply multiply by T. So the corresponding bins are $f_{\rm PSR}T$ and $2f_{\rm PSR}T$.

(f) (3 points) Use the various Fourier theorems and relations that you've seen in class so far to *estimate* (do *not* rigorously derive) how many significant harmonics will be present in the power spectrum if the pulses are roughly Gaussian with fractional width $W/P_{\rm PSR}$.

> **Solution:** The convolution theorem tells us that the Fourier transform of the pulses will be the Fourier transform of a single pulse times the Fourier transform of the Shah function.
>
> We can approximate a single input pulse as a Gaussian with width W in seconds. From the similarity theorem, the Fourier transform of this Gaussian will be another Gaussian with width

$1/W$ Hz. Half of this width will correspond to the negative frequency components, though, and so the positive half (which is what we are concerned with) stretches out to $\sim 1/2W$ Hz.

A train of δ-functions (i.e. the Shah function) separated by $P_{\rm PSR}$ seconds results in a Fourier transform that is also a Shah function, but with the δ-functions separated by $1/P_{\rm PSR} = f_{\rm PSR}$ Hz.

So the number of significant harmonics will be the number of harmonics spaced by $1/P_{\rm PSR}$ Hz that is less than $\sim 1/2W$ Hz, or:

$$\text{number of harmonics} \approx \frac{1/2W}{1/P_{\rm PSR}} = \frac{P_{\rm PSR}}{2W}.$$

6. (4 points) Pulsar timing

(a) (1 point) For a 2 ms pulsar where we can measure pulse times-of-arrival (TOAs) to a fractional precision of 10^{-3} of the pulsar period, estimate the frequency precision we can achieve over a 10-year span of data using pulsar timing.

Solution: In general, the *frequency* f of any signal is just the derivative of its *phase* ϕ with time: $f = d\phi/dt$. TOAs correspond to measurements of the pulse phase. In this case, we can make a fractional phase measurement $\Delta\phi = 10^{-3}$ over a timespan of 10 yrs ($\Delta t = 3.15 \times 10^8$ s). Therefore, the precision is:

$$f_{\rm err} \approx \frac{\Delta\phi}{\Delta t} \approx \frac{1 \times 10^{-3}}{3.15 \times 10^8\,\text{s}} \approx 3 \times 10^{-12}\,\text{Hz}.$$

(b) (1 point) Assume we have a binary MSP with TOAs of precision $\sim 1\,\mu$s that is observed (to get 1 TOA) every 15 days. To approximately what precision can we measure the projected semi-major axis of the orbit $x = a \sin i/c$ (usually measured in lt-sec) after 2 years of observations?

Solution: For a pulsar in an orbit, we simply measure the light travel time across the orbit by seeing how the pulses are delayed (or advanced) in time. Since we can measure a TOA precision of $\sim 1\,\mu$s, that means that the orbital delays can be measured to the same precision (and then changed to length simply by converting seconds into light-seconds).

In this case, though, we are making measurements every 15 days over 2 years, which corresponds to $2 \times 365/15 \sim 49$ independent measurements. Therefore the orbital size can be determined to a precision of

$$\frac{1 \times 10^{-6}\,\text{lt-s}}{\sqrt{49}} \approx 1.4 \times 10^{-7}\,\text{lt-s}.$$

That's about 42 meters!

(c) (2 points) Using Kepler's laws, if the orbital period of the above MSP is 10-days, and assuming that the inclination i is 90°, approximately what is the lowest mass companion star that we can measure using pulsar timing?

Solution: For pulsar and companion masses (m_p and m_c) measured in solar masses, the total orbital separation r ($r = r_p + r_c$) in AU, and the orbital period P in years, Kepler's laws state

$$P^2 = \frac{r^3}{m_p + m_c}.$$

In this case, we can guess that $m_c \ll m_p$ (and so $m_p + m_c \approx m_p$). We can relate r to the masses and the orbital radius r_p (which we know from above) using

$$r_p = r\frac{m_c}{m_p + m_c} \approx r\frac{m_c}{m_p}.$$

Therefore, $r \approx m_p r_p / m_c$ and we can substitute

$$P^2 \approx \frac{(m_p r_p/m_c)^3}{m_p} = \frac{m_p^2 r_p^3}{m_c^3}.$$

Solving for m_c gives us

$$m_c = \left(\frac{m_p^2 r_p^3}{P^2}\right)^{1/3}.$$

Assuming a "normal" neutron star mass of $m_p \sim 1.4\,\mathrm{M}_\odot$, and converting to the appropriate units ($P = 10\,\mathrm{days}$ or $0.0274\,\mathrm{yrs}$, and $r_p = 42\,\mathrm{m}$ or $2.8{\times}10^{-10}\,\mathrm{AU}$) gives us

$$m_c = \left(\frac{(1.4\,\mathrm{M}_\odot)^2(2.8\times10^{-10}\,\mathrm{AU})^3}{(0.0274\,\mathrm{yrs})^2}\right)^{1/3} \approx 4\times10^{-9}\,\mathrm{M}_\odot.$$

That is 0.0013 Earth masses (or about 0.1 Moon masses)! In order to confidently detect such a small signal (and have it believed), a realistic value is probably a factor of ∼10 larger (which would give us at least a 10-σ detection), or about 1 Moon-mass!