Free-free Radio Emission from an HII Region

Thermal bremsstrahlung from an ionized hydrogen cloud (HII region) is often called ***free-free*** emission because it is produced by free electrons scattering off ions without being captured—the electrons are free before the interaction and remain free afterwards. What are the basic properties of free-free radio emission from an astrophysical HII region? Despite all of the simplifications introduced in our description of HII regions, this question isn't like the typical question or problem in a physics course on electromagnetism because it is relatively complicated and "messy," meaning that it can't be solved exactly and various approximations must be made. Consequently our approach will have to be a little different. We can't just write down a few basic equations and solve them analytically for a simple idealized case. A certain amount of astrophysical "intuition" is required. We will have to make judicious approximations to simplify the problem. For example, we will assume that the energy lost by an electron when it interacts with an ion is much smaller than the initial electron energy. We will ignore radiation from electron-electron collisions and from ions. We will encounter integrals over impact parameters that formally diverge and come up with physical reasons to limit the range of integration and still get reasonably accurate, but not exact, results. Most astrophysical conditions are so far removed from personal experience (How hot does 10^4 K feel? How big is a parsec compared with the distance I drive to work in the morning? How much is $M_\odot \approx 2 \times 10^{33}$ g compared with my mass?) that astrophysical intuition depends on keeping track of numerical values for the most important parameters describing the situation, so that it is possible to decide what is important and what can be neglected. The following analysis of free-free emission from an HII region provides a good example of the approach needed to solve many astrophysical problems.

Why should an HII region emit radio radiation at all? Because charged particles are being accelerated electrostatically, and free accelerated charges radiate according to Larmor's formula. Recall that only the component of the electric field perpendicular to the line-of-sight contributes to the electromagnetic radiation at large distances r:

$$E_\perp = \frac{q\dot{v}\sin\theta}{rc^2},$$

where q is the charge, θ is the angle between the acceleration $\dot{v}$ and the line-of-sight to the observer, and c is the speed of light.

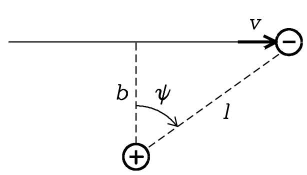
All manner of electrostatic interactions between various charged particles take place in an HII region, but most do not emit significant amounts of radiation. The magnitude of the acceleration is equal to the Coulomb force

$$F = \frac{q_1 q_2}{r_{12}^2} = m\dot{v}$$

divided by the particle mass m. The lightest ion is the hydrogen ion. Its mass is the ***proton mass*** $m_{\rm p} \approx 1.66 \times 10^{-24}$ g), which is about two thousand times the ***electron mass*** $m_{\rm e} \approx 9.11 \times 10^{-28}$ g. In an electron-ion collision, the electron will therefore radiate at least $(m_{\rm p}/m_{\rm e})^2 \approx 4 \times 10^6$ as much power as the ion, so we may neglect the radiation from all of the ions. Interactions between identical particles also do not radiate significantly because the accelerations of the two particles are equal in magnitude but opposite in direction: $\dot{v_1} = -\dot{v_2}$. Thus their radiated electric fields are equal in magnitude but opposite in sign; the net $E_\perp$ approaches zero at large distances where $r \approx r_1 \approx r_2$. We can therefore ignore the radiation from electron-electron collisions. *Only electron-ion collisions are important, and only the electrons radiate significantly.*

Radio Radiation From a Single Electron-ion Interaction

Consider the radiation from an electron passing by a much more massive ion of charge Ze, where $Z = 1$ for a singly ionized atom like hydrogen. Each electron-ion interaction will generate a single pulse of radiation. We will calculate the total energy emitted and the (approximate) frequency spectrum of the pulse. The exact spectrum of an individual pulse will not be needed because the broad distribution of electron energies and impact parameters will smear out such detail in the total spectrum of an HII region.



A light, fast electron passing by a slow, heavy ion. Low-energy radio photons are produced by weak scattering in which the velocity vector $\vec{v}$ of the electron changes little. The distance of closest approach is called the impact parameter b and the interval $\tau = b/v$ is called the collision time.

Radio photons are produced by *weak* interactions, meaning the change ΔE_e of electron kinetic energy is much smaller than the intial kinetic energy E_e. The reason is that the energy $E_\gamma = h\nu$ of a radio photon is much smaller than the average kinetic energy of an electron in an HII region. This numerical comparison is an example of how astrophysical "intuition" simplifies the electron-ion scattering problem.

The mean electron energy in a plasma of temperature T is

$$E_\mathrm{e} = \frac{3kT}{2}$$

For an HII region with $T \sim 10^4$ K, this is

$$E_\mathrm{e} \approx \frac{3 \times 1.38 \times 10^{-16} \text{ erg K}^{-1} \times 10^4 \text{ K}}{2} \approx 2 \times 10^{-12} \text{ erg } \approx 1 \text{ eV}.$$

[This is another useful conversion factor to remember: 1 eV is the typical energy associated with the temperature $T \approx 10^4$ K.] The energy of a photon is $E_\gamma = h\nu$. For example, a radio photon of frequency $\nu = 10$ GHz has energy

$$E_\gamma \approx 6.63 \times 10^{-27} \text{ erg s } \times 10^{10} \text{ Hz } \approx 6.63 \times 10^{-17} \text{ erg } \approx 4 \times 10^{-5} \text{ eV}.$$

The great inequality $E_\gamma \ll E_\mathrm{e}$ means that most radio photons are produced by weak interactions that cause the trajectory of the electron to deflect by only a small angle ($\ll 1$ radian). We may make the approximation that *the electron path is nearly straight as it interacts with an ion to produce radio radiation*.

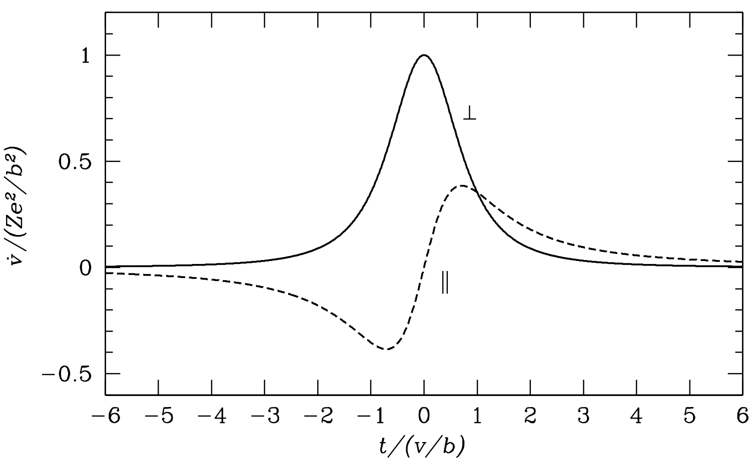
During the interaction, the electron will be accelerated electrostatically both parallel to and perpendicular to its nearly straight path:

$$F_{\|} = m_e \dot{v}_{\|} = \frac{Ze^2}{l^2} \sin\psi = \frac{Ze^2 \sin\psi \cos^2\psi}{b^2}$$

$$F_{\perp} = m_e \dot{v}_{\perp} = \frac{Ze^2}{l^2} \cos\psi = \frac{Ze^2 \cos^3\psi}{b^2},$$

where $\cos\psi = b/l$ and b is the ***impact parameter*** of the interaction, the minimum value of the distance l between the electron and the ion.

For a given impact parameter b, these two equations can be solved to show that the maximum of $\dot{v}_{\|}$ is about 0.38 $\times$ the maximum of $\dot{v}_{\perp}$. Even so, the radio radiation arising from $\dot{v}_{\|}$ is completely negligible. Plotting the variation with time of $\dot{v}_{\|}$ and $\dot{v}_{\perp}$ during the interaction shows pulses with quite different shapes:



The acceleration of an electron by an ion may be resolved into components perpendicular ($\perp$) to and parallel ($\|$) to the electron's velocity. The perpendicular acceleration yields a roughly Gaussian pulse whose power spectrum is dominated by low (radio) frequencies. The parallel acceleration is roughly sinusoidal with no "DC" component, so the resulting pulse is strongest at higher (infrared) frequencies and can be ignored at radio frequencies.

The pulse duration is comparable with the collision time $\tau \approx b/v$. The $\dot{v}_{\|}$ pulse is roughly a sine wave of angular frequency $\omega \sim \tau^{-1} \approx v/b$, which we will soon show is *much* higher than radio frequencies for all relevant impact parameters b. That is, *the parallel acceleration produces infrared radiation but not radio radiation*. The $\dot{v}_{\perp}$ pulse is a single peak whose frequency spectrum extends from zero up to $\sim v/b$ (recall that the Fourier transform of a Gaussian is a Gaussian), so it is rich in radio frequencies.

Considering only the acceleration perpendicular to the electron velocity, we get the instantaneous power emitted from Larmor's formula:

$$P = \frac{2}{3}\frac{e^2 \dot{v}_\perp^2}{c^3} = \frac{2e^2}{3c^3}\frac{Z^2 e^4}{m_e^2}\left(\frac{\cos^3\psi}{b^2}\right)^2$$

The total energy W emitted by the pulse is

$$W = \int_{-\infty}^{\infty} P\,dt\ .$$

Since $(\Delta E_e)/E_e = E_\gamma/E_e \ll 1$, we can make the approximation that *the electron velocity is nearly constant*. From the interaction diagram we see that

$$v = \frac{dx}{dt} \quad \text{and} \quad \tan\psi = \frac{x}{b}$$

so

$$v = \frac{b\,d\tan\psi}{dt} = \frac{b\sec^2\psi\,d\psi}{dt} = \frac{b\ d\psi}{\cos^2\psi\ dt}$$

and

$$dt = \frac{b}{v}\frac{d\psi}{\cos^2\psi}\ .$$

The total pulse energy becomes

$$W = \int_{-\infty}^{\infty} P\,dt = \frac{2}{3}\frac{Z^2 e^6}{c^3 m_e^2 b^4}\int_{-\infty}^{\infty}\cos^6\psi\ dt\ .$$

By changing the variable of integration from t to ψ and then invoking symmetry, we get

$$W = \frac{2}{3}\frac{Z^2 e^6}{c^3 m_e^2 b^4}\int_{-\pi/2}^{\pi/2}\frac{b}{v}\frac{\cos^6\psi}{\cos^2\psi}\ d\psi = \frac{4}{3}\frac{Z^2 e^6}{c^3 m_e^2 b^3 v}\int_0^{\pi/2}\cos^4\psi\,d\psi$$

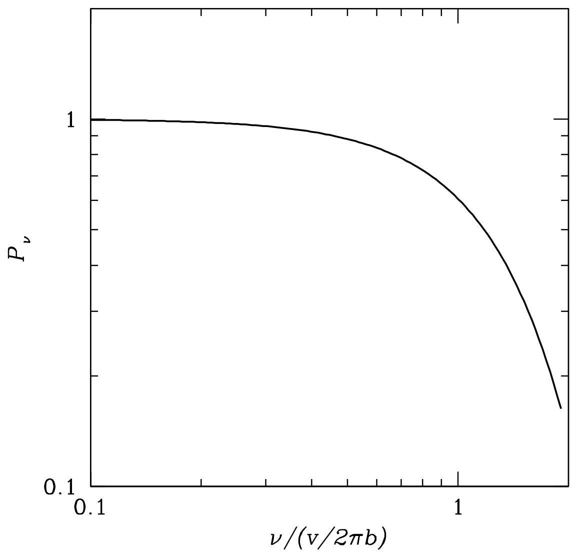
Evaluating the integral yields

$$\int_0^{\pi/2}\cos^4\psi\,d\psi = \frac{3\pi}{16}$$

so the energy radiated by a single electron-ion interaction characterized by impact parameter b and velocity v is

$$W = \frac{\pi Z^2 e^6}{4c^3 m_e^2}\left(\frac{1}{b^3 v}\right)$$

This energy is emitted in a single pulse of duration $\tau \approx b/v$, so the pulse ***power spectrum*** (the spectrum of the *squared* Fourier components) is nearly flat over all frequencies $\nu < \nu_{\rm max} \approx (2\pi\tau)^{-1} \approx v/(2\pi b)$ and falls rapidly at higher frequences. (Recall that the Fourier transform of a Gaussian is a Gaussian.) In principle we could calculate the actual Fourier transform of the pulse shape, but that would only introduce an unnecessary complication into an already complicated calculation. The ranges of velocities v and impact parameters b characterizing electron-ion interactions in an HII region are so wide that averaging over all collision parameters will wash out any fine detail in the spectrum associated with a particular v and b.



The power spectrum of the electromagnetic pulse generated by one electron-ion interaction is nearly flat up to frequency $\nu \approx v/(2\pi b)$, where v is the electron speed and b is the impact parameter, and declines at higher frequencies.

The cutoff frequency $\nu_{\rm max}$ is much higher than any radio frequency; it corresponds to infrared radiation. As we will soon calculate, the typical electron speed in a $T \sim 10^4$ K HII region is $v \approx 7 \times 10^7$ cm s^{-1} and the typical impact parameter is $b \approx 10^{-7}$ cm, so $\nu_{\rm max} \approx 10^{14}$ Hz, corresponding to the near-infrared wavelength $\lambda_{\rm min} \approx 3\,\mu$m.

We make the approximation that the power spectrum shown above is flat out to $\nu = \nu_{\rm max}$ and zero at higher frequencies. Then the average energy *per unit frequency* emitted during a single interaction is approximately

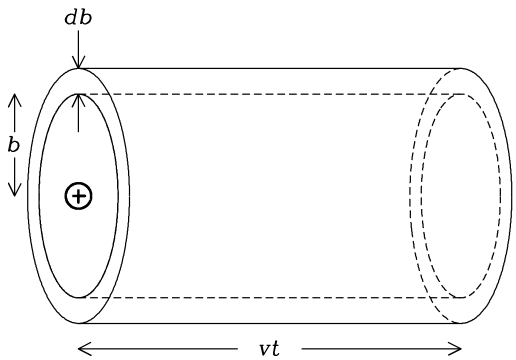
$$W_\nu \approx \frac{W}{\nu_{\rm max}} = \left(\frac{\pi Z^2 e^6}{4 c^3 m_{\rm e}^2 b^3 v}\right)\left(\frac{2\pi b}{v}\right)$$

$$W_\nu \approx \frac{\pi^2}{2}\,\frac{Z^2 e^6}{c^3 m_{\rm e}^2}\left(\frac{1}{b^2 v^2}\right), \quad \nu < \nu_{\rm max} \approx \frac{v}{2\pi b} \approx 10^{14}\ {\rm Hz}$$

Radio Radiation From an HII Region

Next we need to find the distributions of v and b to evaluate the radio emission from an HII region. The distribution of v depends on the electron temperature T. The distribution of b depends on the electron number density N_e (cm^{-3}) and on the ion number density N_I (cm^{-3}).

In LTE, the average kinetic energies of electrons and ions are equal. Since the electrons are much less massive, their speeds are much higher, and we can make the approximation that *the ions are nearly stationary during an interaction*.



The number of electrons with speeds v to $v + dv$ passing by a stationary ion and having impact parameters in the range b to $b + db$ during the time interval t equals the number of electrons with speeds v to $v + dv$ in the cyindrical shell shown here.

Then the number of electrons passing any ion per unit time with impact parameter b to $b + db$ and speed range v to $v + dv$ is

$$N_\mathrm{e}\,(2\pi b\ db)\, v\, f(v)\, dv\ .$$

Here $f(v)$ is the normalized ($\int f(v)dv = 1$) speed distribution of the electrons. The number $N(v, b)$ of such encounters per unit volume per unit time is

$$N(v,b)dv\,db = (2\pi b\ db)[vf(v)\ dv]N_\mathrm{e}N_\mathrm{i}.$$

The spectral power at frequency ν emitted isotropically per unit volume will be $4\pi\epsilon_\nu$, where ϵ_ν is the familiar emission coefficient from radiative transfer. Thus

$$4\pi\epsilon_\nu = \int_{b=0}^{\infty}\int_{v=0}^{\infty} W_\nu(v,b)N(v,b)dv\,db$$

Substituting our results for $W_\nu(v, b)$ and $N(v, b)$ gives

$$4\pi\epsilon_\nu = \int_{b=0}^{\infty}\int_{v=0}^{\infty}\left(\frac{\pi^2 Z^2 e^6}{2c^3 m_\mathrm{e}^2 b^2 v^2}\right)2\pi b\ db\ N_\mathrm{e}N_\mathrm{i}vf(v)\ dv$$

$$4\pi\epsilon_\nu = \frac{\pi^3 Z^2 e^6 N_\mathrm{e}N_\mathrm{i}}{c^3 m_\mathrm{e}^2}\int_{v=0}^{\infty}\frac{f(v)}{v}\ dv\int_{b=0}^{\infty}\frac{db}{b}$$

You can see right away that we have run into a problem: the integral

$$\int_{b=0}^{\infty} \frac{db}{b}$$

diverges logarithmically. There must be some physical limits on the range of the impact parameter b that prevent this divergence. For the time being we will just call those limits $b_{\rm min}$ and $b_{\rm max}$; later we will identify their physical meanings and evaluate them numerically. Meanwhile,

$$4\pi\epsilon_\nu = \frac{\pi^3 Z^2 e^6 N_{\rm e} N_{\rm i}}{c^3 m_{\rm e}^2} \int_{v=0}^{\infty} \frac{f(v)}{v}\, dv \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b}$$

The distribution $f(v)$ of electron speeds in LTE is the ***nonrelativistic Maxwellian distribution***

$$f(v) = \frac{4v^2}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) \qquad (4B1)$$

(see the derivation here or in any thermodynamics textbook). Having $f(v)$ we can now evaluate the integral over the electron speeds:

$$\int_{v=0}^{\infty} \frac{f(v)}{v}\, dv = \frac{4}{\sqrt{\pi}} \left(\frac{m_{\rm e}}{2kT}\right)^{3/2} \int_{v=0}^{\infty} v \exp\left(-\frac{m_{\rm e} v^2}{2kT}\right) dv.$$

Let $u \equiv m_{\rm e} v^2/(2kT)$ so $du = m_{\rm e} v\, dv/(kT)$:

$$\int_{v=0}^{\infty} \frac{f(v)}{v}\, dv = \frac{4}{\sqrt{\pi}} \left(\frac{m_{\rm e}}{2kT}\right)^{3/2} \int_{u=0}^{\infty} \frac{kT}{m_{\rm e}} e^{-u}\, du = \frac{4}{\sqrt{\pi}2} \left(\frac{m_{\rm e}}{2kT}\right)^{1/2} \int_{u=0}^{\infty} e^{-u}\, du$$

$$\int_{v=0}^{\infty} \frac{f(v)}{v}\, dv = \left(\frac{2m_{\rm e}}{\pi kT}\right)^{1/2}$$

In conclusion, we have the following equation for the ***free-free emission coefficient***:

$$\epsilon_\nu = \frac{\pi^2 Z^2 e^6 N_{\rm e} N_{\rm i}}{4 c^3 m_{\rm e}^2} \left(\frac{2m_{\rm e}}{\pi kT}\right)^{1/2} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \qquad (4B2)$$

The next problem is to estimate the minimum and maximum impact parameters $b_{\rm min}$ and $b_{\rm max}$. Since only the logarithms of both parameters appear in Equation 4B2, these estimates don't have to be precise.

To estimate the minimum impact parameter $b_{\rm min}$, we note that the net momentum impulse

$$\Delta P = \int_{-\infty}^{\infty} f dt = \int_{-\infty}^{\infty} eEdt$$

comes from $E_\perp = E\cos\psi$ only because the contribution from $E_\parallel$ is antisymmetric about $t = 0$. Thus

$$\Delta P = \int_{-\infty}^{\infty} e \frac{Ze\cos\psi}{l^2} dt = Ze^2 \int_{-\infty}^{\infty} \frac{\cos^3\psi}{b^2} dt$$

Changing the variable of integration from t to ψ using

$$dt = \frac{b}{v} d\tan\psi = \frac{b}{v}\sec^2\psi d\psi$$

gives

$$\Delta P = \frac{Ze^2}{bv}\int_{-\pi/2}^{\pi/2}\cos\psi d\psi = \frac{2Ze^2}{bv}$$

The maximum possible momentum transfer during the interaction is twice the initial momentum $m_e v$ of the electron, so

$$\boxed{b_{\min} \approx \frac{Ze^2}{m_e v^2}} \qquad (4B3)$$

This result is based on our purely classical treatment of the interaction (See also Jackson's *Classical Electrodynamics*, section 13.1 and problem 13.1, for a more detailed discussion). The uncertainty principle ($\Delta x \Delta p \simeq \hbar$) $does imply a quantum-mechanical limit

$$b_{\min} = \frac{\hbar}{m_e v} ,$$

but this lower limit is generally smaller than the classical limit in HII regions and hence may be ignored. To demonstrate this, we define η as the ratio of the classical to quantum limits

$$\eta = \frac{Ze^2}{3kT} / \frac{\hbar}{m_e v} = \frac{Ze^2}{\hbar v}$$

For an HII region with $T \approx 10^4$, $v \approx (3kT/m_e)^{1/2}$ and $\eta \approx 3$, so the classical limit applies. Only for unusually cold plasmas, $T \ll 1000$ K, would the quantum-mechanical limit be important.

There are two effects that might determine the upper limit $b_{\max}$ to the impact parameter. Because electrostatic forces always dominate gravity on small scales, electrons in the vicinity of a nearly stationary ion are free to rearrange themselves to neutralize, or shield, the ionic charge. The characteristic scale length of this shielding is called the ***Debye length***. From Jackson's *Classical Electrodynamics*, the Debye length is roughly

$$D \approx \left(\frac{kT}{4\pi N_e e^2}\right)^{1/2}$$

The Debye length is quite large in the low-density plasma of a typical HII region. For example, if $T \approx 10^4$ K and $N_e \approx 10^3$ cm^{-3},

$$D \approx \left[\frac{1.38 \times 10^{-16} \text{ erg K}^{-1} \times 10^4 \text{ K}}{4\pi \times 10^3 \text{ cm}^{-3} \times (4.8 \times 10^{-10} \text{ statcoulomb})^2} \right]^{1/2} \approx 22 \text{ cm}.$$

An independent upper limit to the impact parameter is the largest value of b that can generate a significant amount of power at some relevant radio frequency ν. Recall that the pulse power per unit bandwidth is small above angular frequencies

$$\omega \approx \frac{v}{b}$$

so

$$b_{\max} \approx \frac{v}{\omega} = \frac{v}{2\pi\nu},$$

where ν is the radio frequency being considered.

Clearly, the *relevant* upper limit $b_{\max}$ in any particular situation is the *smaller* of these two upper limits. To see which is smaller, we calculate the limits for a typical HII region and a typical radio frequency.

Example: Estimate $b_{\min}$ and $b_{\max}$ for a pure HII region ($Z = 1$) with $T \approx 10^4$ K observed at $\nu = 10$ GHz $= 10^{10}$ Hz.

$$b_{\min} \approx \frac{Ze^2}{m_e v^2} \approx \frac{Ze^2}{3kT}$$

$$b_{\min} \approx \frac{(4.8 \times 10^{-10} \text{ statcoulomb})^2}{3 \times 1.38 \times 10^{-16} \text{ erg K}^{-1} \times 10^4 \text{ K}} \approx 5.6 \times 10^{-8} \text{ cm}$$

$$b_{\max} \approx \frac{v}{2\pi\nu} \approx \left(\frac{3kT}{m_e} \right)^{1/2} / (2\pi\nu)$$

so the largest impact parameter giving significant power at $\nu = 10^{10}$ Hz is

$$b_{\max} \approx \left(\frac{3 \times 1.38 \times 10^{-16} \text{ erg K}^{-1} \times 10^4 \text{ K}}{9.1 \times 10^{-28} \text{ g}} \right)^{1/2} / (2\pi \times 10^{10} \text{s}^{-1}) \approx 1.1 \times 10^{-3} \text{ cm} \; .$$

The maximum impact parameter capable of generating power at this frequency is much smaller than the Debye length in an HII region, so the Debye length is irrelevant. An electron takes so long to move a $D \approx 22$ cm Debye length that it would emit at unobservably low frequencies $\nu < 1$ MHz. The Debye length becomes relevant only in much denser plasmas such as the solar chromosphere ($N_e \approx 10^{12}$ cm^{-3}).

Our simple estimate of the ratio

$$\frac{b_{\max}}{b_{\min}} \approx \left(\frac{3kT}{m_e} \right)^{1/2} (2\pi\nu)^{-1} \left(\frac{3kT}{Ze^2} \right) \approx \left(\frac{3kT}{m_e} \right)^{3/2} \frac{m_e}{2\pi Z e^2 \nu}$$

is very close to the result of Oster's (1961, Rev. Modern Physics, 33, 525) very detailed derivation. The ratio $(b_{\rm max}/b_{\rm min})$ is roughly 10^4, which is much greater than the fractional velocity range $\sigma_v/v \approx 1$ in the Maxwellian velocity distribution. Also note that

$$\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \sim 10$$

varies slowly with changes in either $b_{\rm max}$ or $b_{\rm min}$, so small uncertainties in these limits have little effect on the calculated emission coefficient of an HII region.

Since the HII region is in local thermodynamic equilibrium (LTE) at some temperature T, we can use Kirchoff's law to calculate the absorption coefficient from the emission coefficient and the blackbody brightness law:

$$\kappa_\nu = \frac{\epsilon_\nu}{B_\nu(T)} \ .$$

In the Rayleigh-Jeans limit,

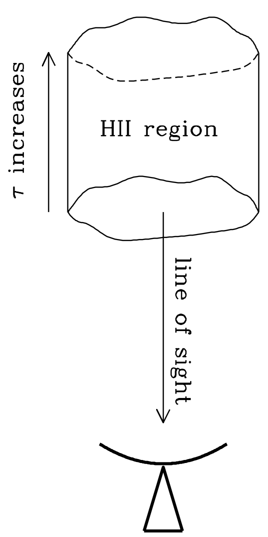
$$\kappa_\nu = \frac{\epsilon_\nu c^2}{2kT\nu^2}$$

and

$$\kappa_\nu = \frac{1}{\nu^2 T^{3/2}}\left[\frac{Z^2 e^6}{c} N_{\rm e} N_{\rm i} \frac{1}{\sqrt{2\pi (m_{\rm e} k)^3}}\right] \frac{\pi^2}{4} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \qquad (4B4)$$

Because the $b_{\rm max}$ does vary slowly with frequency, the absorption coefficient is not exactly proportional to ν^{-2}. A good numerical approximation is $\kappa_\nu \propto \nu^{-2.1}$.

The total opacity τ_ν of an HII region is the integral of $-\kappa_\nu$ along the line of sight, as illustrated below.



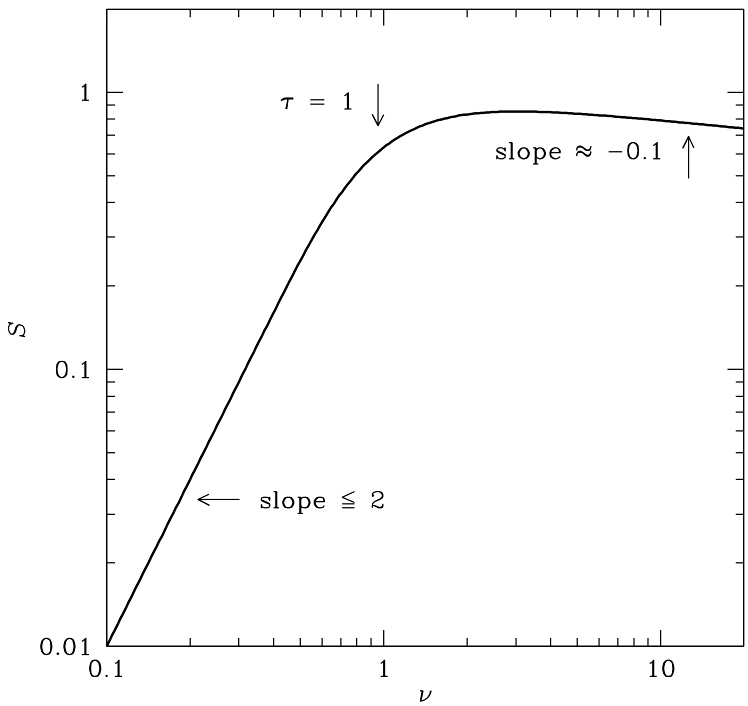
Astronomers often approximate HII regions by uniform cylinders whose axis is the line of sight *because this gross oversimplification finesses the radiative-transfer problem. It is for good reason that astronomers are frequently the butt of jokes beginning "Consider a spherical cow...".*

$$\tau_\nu = \int_{\rm los} -\kappa_\nu \, ds \propto \int \frac{N_{\rm e} N_{\rm i}}{\nu^{2.1} T^{3/2}} ds \approx \int \frac{N_{\rm e}^2}{\nu^{2.1} T^{3/2}} ds$$

At frequencies low enough that $\tau_\nu \gg 1$, the HII region becomes opaque, its spectrum approaches that of a black body with temperature $T \approx T_{\rm e} \sim 10^4$ K, and its flux density varies as $S \propto \nu^2$. At very high frequencies, $\tau_\nu \ll 1$, the HII region is nearly transparent, and

$$S_\nu \propto \frac{2kT\nu^2}{c^2} \tau_\nu \propto \nu^{-0.1}$$

On a log-log plot, the overall spectrum of a uniform HII region looks like this, with the spectral break corresponding to the frequency at which $\tau_\nu \approx 1$:



The radio spectrum of our idealized HII region. It is a black body at low frequencies, with slope 2 if a uniform cylinder and <2 otherwise. At some frequency the optical depth $\tau_\nu = 1$, and at much higher frequencies the spectral slope becomes ≈ -0.1 because the opacity coefficient $\kappa_\nu \propto \nu^{-2.1}$. The brightness at low frequencies depends only on the electron temperature. The brightness at high frequencies depends also on the emission measure (defined below) of the HII region.

The spectral slope on a log-log plot is often called the ***spectral index*** and denoted by α, which is defined ambiguously:

$$\alpha \equiv \pm \frac{d \log S}{d \log \nu}$$

Beware the $\pm$ *sign!* Unfortunately both sign conventions are found in the literature, and you have to look carefully at each paper to find out which one is being used. With the $+$ sign convention, the low-frequency spectral index of a uniform HII region would be $\alpha = +2$. The $-$ sign convention was introduced in the early days of radio astronomy because most sources discovered at low frequencies are stronger at low frequencies than at high frequencies. Thus $\alpha = +0.7$ might mean

$$\frac{d \log S}{d \log \nu} = -0.7$$

The $(+)$ spectral index of any inhomogeneous HII region will be $\alpha \approx = -0.1$ well above the break frequency, but the break will be more gradual and the low-frequency slope will be somewhat less than $+2$ just below the break. For example, ionized winds from stars are quite inhomogeneous. Mass conservation in a constant-velocity, isothermal spherical wind implies that the electron density is inversely proportional to the square of the distance from the star: $N_{\rm e} \propto r^{-2}$. The low-frequency spectral index of free-free emission by such a wind is closer to $+0.6$ than to $+2$.

In the astronomical literature you will encounter the term ***emission measure*** (EM) defined by the integral of $N_{\rm e}^2$ along the line-of-sight and usually expressed in astronomically convenient units:

$$\frac{EM}{\rm pc\ cm^{-6}} \equiv \int_{\rm los} \left(\frac{N_e}{\rm cm^{-3}} \right)^2 d\left(\frac{s}{\rm pc} \right) \qquad (4B5)$$

Because κ_ν is proportional to $N_{\rm e} N_{\rm i} \approx N_{\rm e}^2$, the optical depth τ is proportional to the emission measure. The emission measure is commonly used to parameterize τ in astronomically convenient units:

$$\tau_\nu \approx 3.014 \times 10^{-2} \left(\frac{T_{\rm e}}{\rm K} \right)^{-3/2} \left(\frac{\nu}{\rm GHz} \right)^{-2} \left(\frac{EM}{\rm pc\ cm^{-6}} \right) \langle g_{\rm ff} \rangle \ ,$$

where the ***free-free Gaunt factor*** $\langle g_{\rm ff} \rangle$ is a parameter that absorbs the weak frequency dependence associated with the logarithmic term in κ_ν:

$$\langle g_{\rm ff} \rangle \approx \ln \left[4.955 \times 10^{-2} \left(\frac{\nu}{\rm GHz} \right)^{-1} \right] + 1.5 \ln \left(\frac{T_{\rm e}}{\rm K} \right)$$

We end up with a very good approximation for ***free-free opacity*** that is easy to evaluate numerically:

$$\tau_\nu \approx 3.28 \times 10^{-7} \left(\frac{T_e}{10^4 \text{ K}} \right)^{-1.35} \left(\frac{\nu}{\text{GHz}} \right)^{-2.1} \left(\frac{EM}{\text{pc cm}^{-6}} \right) \qquad \text{(4B6)}$$

From the optical depth τ and the electron temperature T_e we can calculate the brightness temperature

$$T_b = T_e (1 - e^{-\tau}) \ .$$

We normally don't know the structure of the HII region along the line-of-sight, so it is common to approximate the geometry of an HII region by a circular cylinder whose axis lies along the line-of-sight, and whose axis length equals its diameter. We also suppose that the temperature and density are constant throughout this volume. Then it is very easy to estimate physical parameters of the HII region (e.g., electron density, temperature, emission measure, production rate N_{Ly} of ionizing photons) from the observed radio spectrum, once the distance to the HII region is known.

A useful approximation relating the production rate of ionizing photons to the ***free-free spectral luminosity*** L_ν at the high frequencies where $\tau \ll 1$ of an HII region in ionization equilibrium is

$$\left(\frac{N_{\text{Ly}}}{\text{s}^{-1}} \right) \approx 6.3 \times 10^{52} \left(\frac{T_e}{10^4 \text{ K}} \right)^{-0.45} \left(\frac{\nu}{\text{GHz}} \right)^{0.1} \left(\frac{L_\nu}{10^{20} \text{ W Hz}^{-1}} \right) \qquad \text{(4B7)}$$

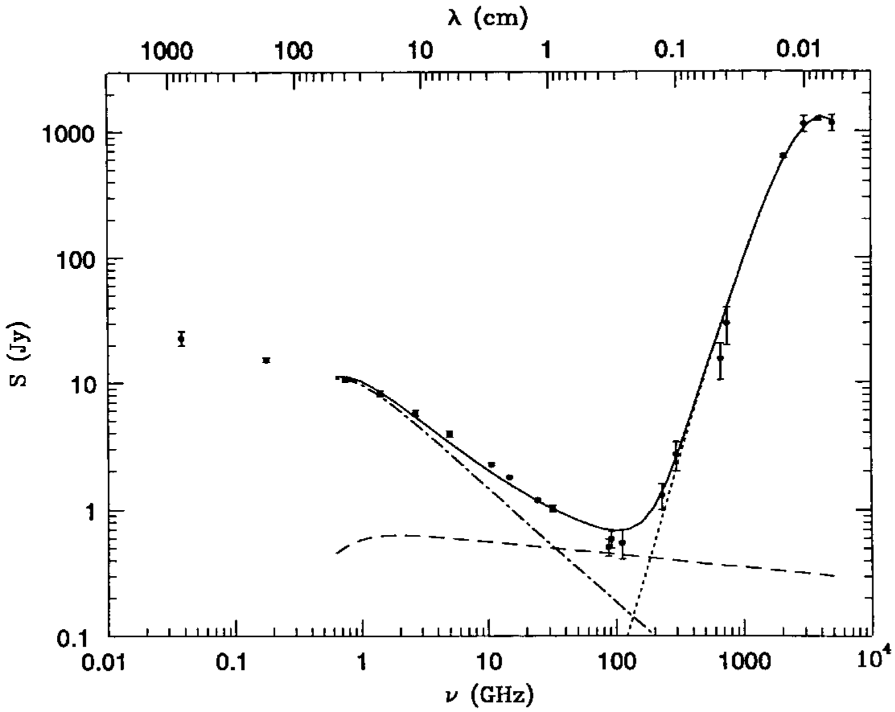
Example: The interstellar medium of our Galaxy contains a diffuse ionized component, some of which is "warm" ($T_e \approx 10^4$ K) and some is "hot" ($T_e \approx 10^6$ K). These two *phases* are roughly in pressure equilibrium so the hot medium is less dense by a factor of $\sim 10^2$. The combination of high T_e and low N_e of the hot phase means that only the warm component contributes significantly to the free-free opacity of the ISM. The warm ionized gas is largely confined to the disk of our Galaxy, where we reside. There must be some frequency ν below which this disk becomes opaque and we cannot see out of our Galaxy, even in the direction perpendicular to the disk.

From the observed brightness spectrum in the direction perpendicular to the disk, Cane (1979, MNRAS, 189, 465) found that $\tau \approx 1$ at $\nu \approx 3$ MHz. We can use this result to estimate the typical electron density in the warm ISM.

$$1 \approx 3.28 \times 10^{-7} \times (1)^{-1.35} \times 0.003^{-2.1} \times \langle N_e^2 \rangle \times 1000 \text{ pc}$$

$$\langle N_e^2 \rangle^{1/2} \approx \left(\frac{0.003^{2.1}}{3.28 \times 10^{-4}} \right)^{1/2} \approx 0.1 \text{ cm}^{-3}$$

Free-free emission accounts for about 10% of the 1 GHz continuum luminosity in most spiral galaxies. It is the strongest component in the frequency range from $\nu \approx 30$ GHz to $\nu \approx 200$ GHz, above which thermal emission from cool dust grains dominates. Free-free absorption flattens the low-frequency spectra of spiral galaxies, and the frequency at which $\tau \approx 1$ is higher in galaxies with high star-formation rates, especially if the star formation is confined to a compact region near the nucleus.



The radio and far-infrared spectrum of the nearby starburst galaxy M82. The contribution of free-free emission is indicated by the nearly horizontal dashed line. Synchrotron radiation and thermal dust emission dominate at low and high frequencies, respectively. Free-free absorption from HII regions distributed throughout the galaxy absorbs some of the synchrotron radiation and flattens the overall spectrum at the lowest frequencies.