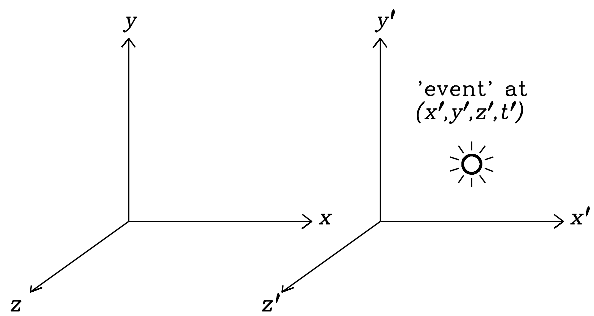
Synchrotron Power

Cosmic rays are astrophysical particles (electrons, protons, and heavier nuclei) with extremely high energies. Cosmic-ray electrons in the galactic magnetic field emit the ***synchrotron radiation*** that accounts for most of the continuum emission from our Galaxy at frequencies below about 30 GHz. We can use Larmor's formula to calculate the synchrotron power and synchrotron spectrum of a single electron in an inertial frame in which the electron is instantaneously at rest, but we need the ***Lorentz transform*** of special relativity to transform these results to the frame of an observer at rest in the Galaxy.



The Lorentz transforms relating the ***event*** coordinates (x, y, z, t) in the unprimed frame and the coordinates (x', y', z', t') in the primed frame moving with velocity v in the x direction are:

$$x = \gamma(x' + vt') \qquad y = y' \qquad z = z' \qquad t = \gamma(t' + \beta x'/c) \tag{5B1}$$

$$x' = \gamma(x - vt) \qquad y' = y \qquad z' = z \qquad t' = \gamma(t - \beta x/c) \tag{5B2}$$

where

$$\beta \equiv v/c \tag{5B3}$$

and

$$\gamma \equiv (1 - \beta^2)^{-1/2} \tag{5B4}$$

is called the ***Lorentz factor***.

If $(\Delta x, \Delta y, \Delta z, \Delta t)$ and $(\Delta x', \Delta y', \Delta z', \Delta t')$ are the coordinate differences between two events, the differential form of the (linear) Lorentz transforms is:

$$\Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta y = \Delta y' \qquad \Delta z = \Delta z' \qquad \Delta t = \gamma(\Delta t' + \beta \Delta x'/c) \qquad \text{(5B}$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta y' = \Delta y \qquad \Delta z' = \Delta z \qquad \Delta t' = \gamma(\Delta t - \beta \Delta x/c) \qquad \text{(5B}$$

Here is a derivation of these results that you should review before proceeding.

Using the famous equation $E = mc^2$ we can calculate the energy equivalent to the rest mass m_e of an electron:

$$E = m_e c^2 = 9.1 \times 10^{-28}\ \mathrm{g} \times (3 \times 10^{10}\ \mathrm{cm\ s}^{-1})^2 = 8.2 \times 10^{-7}\ \mathrm{erg}$$

$$E = \frac{8.2 \times 10^{-7}\ \mathrm{erg}}{1.60 \times 10^{-12}\ \mathrm{erg\ (eV)}^{-1}} = 5.1 \times 10^5\ \mathrm{eV} = 0.51\ \mathrm{MeV}$$

Cosmic-ray electrons with energies in the range 10^9 to 10^{14} eV have

$$\gamma \approx \frac{10^9 \text{ to } 10^{14}}{0.51 \times 10^6} \approx 10^3 \text{ to } 10^8 \gg 1$$

and such cosmic-ray electrons are called ***ultrarelativistic*****.** These electrons still move on spiral paths along magnetic field lines, but the angular frequencies of their orbits are lower because the inertial masses of the electrons are higher by a factor of γ:

$$\omega_B = \frac{eB}{\gamma m_e c} = \frac{\omega_G}{\gamma}$$

Example: A cosmic-ray electron with $\gamma = 10^5$ in the Galactic magnetic field $B \approx 5 \times 10^{-6}$ G will have an orbital frequency

$$\nu_B \equiv \frac{\omega_B}{2\pi} \approx 14 \times 10^{-5}\ \mathrm{Hz} \approx 1 \text{ cycle in two hours.}$$

Since $v \approx c$ whenever $\gamma \gg 1$, the orbital radius R of an ultrarelativistic electron is quite large:

$$R \approx \frac{c}{\omega_B} \approx \frac{3 \times 10^{10}\ \mathrm{cm\ s}^{-1}}{2\pi \times 14 \times 10^{-5}\ \mathrm{Hz}} \approx 3.4 \times 10^{13}\ \mathrm{cm} \approx 2\ \mathrm{AU}$$

At first glance, these results are not very promising for the production of radio radiation: the high relativistic masses of cosmic-ray electrons reduce their orbital frequencies and accelerations to extremely low values. However, the Larmor radiation formula is only valid at low velocities; that is, in inertial frames in which the electron is nearly at rest. In the observer's frame, two relativistic effects account for the strong radio radiation: (1) the total power is multiplied by γ^2 and (2) beaming turns the slow sinusoidal radiation into a series of sharp pulses containing power at much higher frequencies $\sim \gamma^3 \nu_{\rm B} = \gamma^2 \nu_{\rm G}$. We proceed to calculate these relativistic corrections.

Synchrotron Power From a Single Electron

Nonrelativistic equations such as Larmor's equation describing the electromagnetic radiation from an accelerated charge are correct only in inertial frames where the electron velocity $v \ll c$, but the results can be transformed to any other inertial frame by the Lorentz transform. In this way, it is possible to calculate the total power radiated by an ultrarelativistic electron in a magnetic field parallel to the x-axis. We use primed coordinates to describe an inertial frame in which the electron is (temporarily) nearly at rest. Then Larmor's equation correctly gives

$$P' = \frac{2e^2 (a'_\perp)^2}{3c^3} \ .$$

What is $a_\perp$, the magnetic acceleration of the electron in the galaxy frame?

$$v_{\rm y} \equiv \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt} = v'_{\rm y}\frac{dt'}{dt}$$

The differential form of the Lorentz transform yields

$$\frac{dt'}{dt} = \frac{1}{\gamma} \ ,$$

so

$$v_{\rm y} = \frac{v'_{\rm y}}{\gamma} \ .$$

This factor of γ is a consequence of relativistic ***time dilation***—clocks in moving frames appear to run slow by a factor γ. Consequently,

$$a_{\rm y} \equiv \frac{dv_{\rm y}}{dt} = \frac{dv_{\rm y}}{dt'}\frac{dt'}{dt} = \frac{1}{\gamma}\frac{dv'_{\rm y}}{dt'}\frac{dt'}{dt} = \frac{a'_{\rm y}}{\gamma^2} \ .$$

Similarly, $a_{\rm z} = a'_{\rm z}/\gamma^2$ so

$$a_\perp = \frac{a'_\perp}{\gamma^2} \ .$$

Thus

$$P' = \frac{2e^2 (a'_\perp)^2}{3c^3} = \frac{2e^2 a_\perp^2 \gamma^4}{3c^3}$$

How do we transform P' to P, the power measured by an observer at rest in the Galaxy? The following argument is from Rindler's *Essential Relativity*, p. 98. Imagine two identical electrons of rest mass $m_{\rm e}$, one at rest in the unprimed frame and the other at rest in the primed frame. If one electron is slightly displaced from the other along the y-axis, they will interact as they pass each other and be accelerated in the $\pm y$ direction. Observers at rest in each frame see "their" electron move with some small $v_{\rm y} \ll c$, but the "other" electron will appear to move in the opposite y direction by a factor γ more slowly because of time dilation—recall the result $v_{\rm y} = v_{\rm y'}/\gamma$ above. Invoking momentum conservation, observers in each frame conclude that the "other" electron has inertial mass $m_{\rm e}\gamma$ and hence its energy is greater by the same factor γ. Thus

$$P \equiv \frac{dE}{dt} = \frac{dE}{dt'}\frac{dt'}{dt} = \frac{dE}{dE'}\frac{dE'}{dt'}\frac{dt'}{dt} = \gamma P' \frac{1}{\gamma} = P' \ ;$$

that is, *power is the same in all frames*. Consequently,

$$P = \frac{2e^2 a_\perp^2 \gamma 4}{3c^3} \qquad (a_\parallel = 0)$$

Recall that

$$\omega_{\rm B} = \frac{eB}{\gamma mc}$$

and, by force balance,

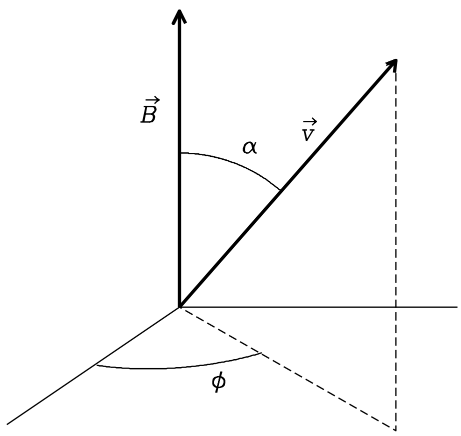
$$a_\perp \equiv \frac{dv_\perp}{dt} = \omega_{\rm B} v_\perp$$

so

$$a_\perp = \frac{eBv_\perp}{\gamma mc} = \frac{eBv \sin\alpha}{\gamma mc} \ ,$$

where the angle α between $\vec{v}$ and $\vec{B}$ is called the ***pitch angle***. For a given pitch angle α, the time-averaged radiated power of a single electron is

$$P = \frac{2e^2}{3c^3}\gamma^2 \frac{e^2 B^2}{m^2 c^2} v^2 \sin^2\alpha$$



The pitch angle α between the directions of the magnetic field $\vec{B}$ and the electron velocity $\vec{v}$.

We can express this power in terms of the ***Thomson cross section*** of an electron, $\sigma_{\rm T}$. The Thomson cross section is the classical scattering cross section for electromagnetic radiation. If a plane wave of electromagnetic radiation is incident on a charge at rest, the electric field of that radiation will accelerate the charge, which in turn will radiate power in other directions according to Larmor's equation. This process is called scattering, not absorption, because the total power in electromagnetic radiation is unchanged: all of the power lost from the incident plane wave is reradiated in other directions. In one of the problem sets, you show that the geometric area that would intercept this amount of incident power is

$$\boxed{\sigma_{\rm T} \equiv \frac{8\pi}{3}\left(\frac{e^2}{m_{\rm e}c^2}\right)^2} \qquad (5B7)$$

Numerically,

$$\sigma_{\rm T} = \frac{8\pi}{3}\left[\frac{(4.8\times10^{-10}\ {\rm statcoul})^2}{9.1\times10^{-28}\ {\rm g}\ (3\times10^{10}\ {\rm cm\ s}^{-1})^2}\right] \approx 6.65\times10^{-25}\ {\rm cm}^2$$

The reason for using the Thomson cross section will become clear when we discuss inverse-Compton scattering of radiation by the same cosmic rays that are producing synchrotron radiation.

Also, we can replace B^2 by the ***magnetic energy density***

$$\boxed{U_{\rm B} = \frac{B^2}{8\pi}} \qquad (5B8)$$

to get

$$P = \left[\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\right] 2\left(\frac{B^2}{8\pi}\right) c\gamma^2 \frac{v^2}{c^2} \sin^2\alpha$$

$$\boxed{P = 2\sigma_{\rm T}\beta^2\gamma^2 c\, U_{\rm B} \sin^2\alpha} \qquad (5B9)$$

where $\beta \equiv v/c$. The radiated power depends only on physical constants, the square of the electron energy (via γ^2; $\beta^2 \approx 1$ for all $\gamma \gg 1$), the magnetic energy density, and the pitch angle.

The relativistic electrons in radio sources can have **lifetimes** of thousands to millions of years before losing their ultrarelativistic energies via synchrotron radiation or other processes, so they are scattered repeatedly by magnetic-field fluctuations and charged particles in their environment, and the distribution of their pitch angles α gradually becomes random. The average synchrotron power $\langle P \rangle$ per electron in an ensemble of electrons with the same Lorentz factor γ but random pitch angles is therefore

$$\langle P \rangle = 2\sigma_{\rm T}\beta^2\gamma^2 c\, U_{\rm B} \langle \sin^2\alpha \rangle \ .$$

$$\langle \sin^2\alpha \rangle \equiv \int \sin^2\alpha\, d\Omega \Big/ \int d\Omega = \frac{1}{4\pi}\int \sin^2\alpha\, d\Omega$$

$$\langle \sin^2\alpha \rangle = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\alpha=0}^{\pi} \sin^2\alpha \sin\alpha \, d\alpha\, d\phi$$

$$\langle \sin^2\alpha \rangle = \frac{1}{4\pi} 2\pi \frac{4}{3} = \frac{2}{3}$$

$$\boxed{\langle P \rangle = \frac{4}{3}\sigma_{\rm T}\beta^2\gamma^2 c U_{\rm B}} \qquad (5B10)$$

This is the ***average synchrotron power*** emitted by a relativistic electron. When $\gamma \gg 1$, $\beta \approx 1$ and the β^2 factor may be ignored. Relativistic effects make the synchrotron power a factor γ^2 larger than in the limit $v \ll c$, so for electrons with $\gamma \sim 10^4$, the power radiated by each electron is multiplied by 10^8, a huge amount.