

# Synchrotron Spectrum

## The Synchrotron Spectrum of a Single Electron

Our next problem is to explain how the synchrotron mechanism can yield radio radiation at frequencies much higher than  $\omega_B = \omega_G/\gamma$ . To solve it, we first calculate the angular distribution of the radiation in the observer's frame.

**Relativistic aberration** causes the Larmor dipole pattern in the electron frame to become beamed sharply in the direction of motion as  $v$  approaches  $c$ . This beaming follows directly from the relativistic velocity equations implied by the differential form of the Lorentz transform. For an electron moving in the  $x$  direction,

$$v_x \equiv \frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt}$$

$$v_x = \gamma \left( \frac{dx'}{dt'} + v \frac{dt'}{dt} \right) \left( \frac{dt}{dt'} \right)^{-1}$$

$$v_x = \gamma(v'_x + v) \left[ \gamma \left( 1 + \frac{\beta}{c} \frac{dx'}{dt'} \right) \right]^{-1}$$

$$v_x = (v'_x + v) \left( 1 + \frac{\beta v'_x}{c} \right)^{-1}.$$

In the  $y$  direction perpendicular to the electron velocity, the velocity formula gives

$$v_y \equiv \frac{dy}{dt} = \frac{dy}{dt'} \frac{dt'}{dt} = \frac{dy'}{dt'} \left( \frac{dt}{dt'} \right)^{-1}$$

$$v_y = \frac{v'_y}{\gamma} \left( 1 + \frac{\beta v'_x}{c} \right)^{-1}$$

Consider the synchrotron photons emitted with speed  $c$  at an angle  $\theta'$  from the  $x'$  axis. Let  $v'_x$  and  $v'_y$  be the projections of the photon speed onto the  $x'$  and  $y'$  axes. Then

$$\cos \theta' = \frac{v'_x}{c} \quad \sin \theta' = \frac{v'_y}{c}$$

In the observer's frame we have for the same photons,

$$\cos \theta = \frac{v_x}{c} \quad \sin \theta = \frac{v_y}{c}$$

Using our equations for velocities, we get

$$\cos \theta = \left( \frac{v'_x + v}{1 + \beta v'_x/c} \right) \frac{1}{c} = \left( \frac{c \cos \theta' + v}{1 + \beta c \cos \theta'/c} \right) \frac{1}{c} = \left( \frac{c \cos \theta' + v}{1 + \beta \cos \theta'} \right) \frac{1}{c}$$

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}$$

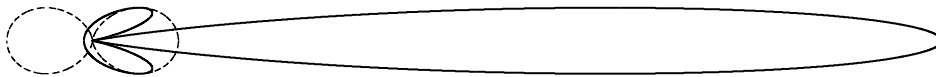
$$\sin \theta = \frac{v'_y}{c\gamma(1 + \beta v'_x/c)}$$

$$\sin \theta = \frac{\sin \theta'}{\gamma(1 + \beta \cos \theta')}$$

In the frame moving with the electron, the Larmor equation implies a power pattern proportional to  $\cos^2 \theta'$  with nulls at  $\theta' = \pm\pi/2$ . In the observer's frame, these nulls appear at much smaller angles

$$\sin \theta = \frac{\sin \theta'}{\gamma(1 + \beta \cos \theta')} \approx \frac{1}{\gamma} \approx \theta$$

since  $1/\gamma \ll 1$ . We see the radiation confined to a very narrow beam of width  $2/\gamma$  between nulls:



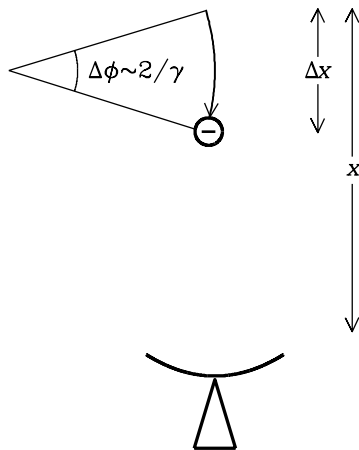
**Relativistic beaming** transforms the dipole pattern of Larmor radiation in the electron frame (dotted curve) to a narrow searchlight beam in the observer's frame. The solid curve shows the observed power pattern for  $\gamma = 5$ . The observed angle between the nulls of the forward beam  $\approx 2/\gamma$ , and the peak gain  $\approx 2\gamma$ .

For example, a 10 GeV electron has  $\gamma \approx 2 \times 10^4$  so  $2/\gamma \approx 10^{-4}$  rad  $\approx 20$  arcsec! The observer sees a short pulse of radiation emitted during only the tiny fraction

$$\frac{2}{2\pi\gamma} = \frac{1}{\pi\gamma}$$

of the electron orbit when the electron is moving directly toward the observer.

The duration  $\Delta t$  of the observed pulse is even shorter than the time the electron needs to cover  $1/(\pi\gamma)$  of its orbit because the electron is visible only while it is moving directly toward the observer with a speed approaching  $c$ . In the observer's frame, the electron is nearly keeping up with the photons that it is emitting.



The beamed radiation from a relativistic electron is visible only while the electron's velocity points within  $\pm 1/\gamma$  of the line-of-sight ( $\Delta\theta \approx 2/\gamma$ ). During that time  $\Delta t$  the electron moves a distance  $\Delta x = v\Delta t$  toward the observer, almost keeping up with the photons that move a distance  $c\Delta t$ . As a result, the observed pulse duration is shortened by a factor  $(1 - v/c)$ .

$$\Delta t = t(\text{end of observed pulse}) - t(\text{start of observed pulse})$$

$$\Delta t = \frac{\Delta x}{v} + \frac{(x - \Delta x)}{c} - \frac{x}{c}$$

The first term in this equation represents the time taken by the electron to cover the distance  $\Delta x$ , the second is the light travel time from the electron position at the *end* of the pulse to the observer, and the third is the light travel time from the electron position at the *beginning* of the pulse to the observer. Note that the duration

$$\Delta t = \frac{\Delta x}{v} - \frac{\Delta x}{c} = \frac{\Delta x}{v} \left(1 - \frac{v}{c}\right) \ll \frac{\Delta x}{v}$$

of the observed pulse is much less than the time it takes the electron to move a distance  $\Delta x$  because, in the observer's frame, the electron nearly keeps up with its radiation. In the limit  $v \rightarrow c$ ,

$$\left(1 - \frac{v}{c}\right) = \left(1 - \frac{v}{c}\right) \frac{1+v/c}{1+v/c} = \frac{1-v^2/c^2}{1+v/c} \approx \frac{\gamma^{-2}}{2} = \frac{1}{2\gamma^2}$$

so

$$\Delta t = \frac{\Delta x}{v} \frac{1}{2\gamma^2} = \frac{\Delta\theta}{\omega_B} \frac{1}{2\gamma^2}.$$

Recall that  $\Delta\theta \approx 2/\gamma$  so

$$\Delta t = \frac{2}{\gamma\omega_B 2\gamma^2} = \frac{1}{\gamma^3\omega_B} = \frac{1}{\gamma^2\omega_G}$$

is the *full* observed duration of a pulse.

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Example: Just how short is the observed duration of one synchrotron pulse? In the typical interstellar magnetic field of our Galaxy,  $B \approx 5 \times 10^{-6}$  G so  $\omega_G \approx 2\pi \times 14$  rad s<sup>-1</sup>. For an electron having  $\gamma \approx 10^4$ ,

$$\Delta t \approx \frac{1}{\gamma^2\omega_G} \approx \frac{1}{(10^4)^2 \times 2\pi \times 14} \approx 10^{-10} \text{ s}$$

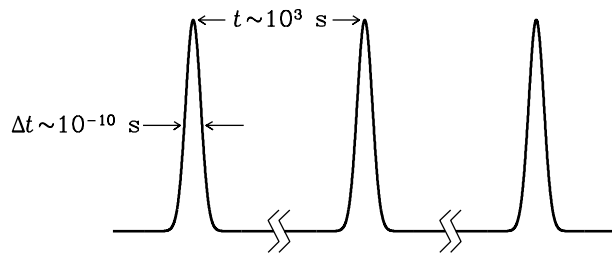
Allowing for the motion of the electron parallel to the magnetic field, we replace the total magnetic field by its perpendicular component  $B \sin \alpha$ , yielding

$$\Delta t = \frac{1}{\gamma^2\omega_G \sin \alpha},$$

where  $\alpha$  is the pitch angle of the electron.

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Thus a plot of the power received as a function of time is very spiky:



Synchrotron radiation is a very spiky series of widely spaced narrow pulses. The numerical values indicated on the plot of power versus time correspond to an electron with  $\gamma \approx 10^4$  in a magnetic field  $B \approx 5\mu\text{G}$  typical of the interstellar medium in our Galaxy.

If  $\gamma \approx 10^4$ , the duration of each pulse is  $\Delta t \sim 10^{-10}$  s and if  $B \approx 5\mu\text{G}$ , the spacing between pulses is  $\gamma/\nu_G \sim 10^3$  s.

The observed synchrotron power spectrum is the Fourier transform of this time series of pulses. Instead of calculating the Fourier transform directly, we note first that the pulse train is the [convolution](#) of the profile  $p(t)$  of an individual pulse with the **Shah function** (see Appendix B of Rohlfs & Wilson or Bracewell's valuable reference book *The Fourier Transform and Its Applications*):

$$III(t/\Delta t) \equiv \sum_{n=-\infty}^{\infty} \delta[(t/\Delta t) - n]$$

where each **delta function**  $\delta$  is an infinitesimally narrow spike at integer  $t/\Delta t = n$  whose integral is unity. Then we use the [convolution theorem](#) to show that the Fourier transform of the pulses is the product of the Fourier transform of one pulse times the Fourier transform of the shah function.

From appendix B of Rohlfs & Wilson, the Fourier transform of the shah function is also the shah function, so by the similarity theorem, the Fourier transform of

$$III\left(\frac{t\nu_G}{\gamma}\right)$$

is proportional to

$$III\left(\frac{\nu\gamma}{\nu_G}\right),$$

a nearly continuous series of spikes in the frequency domain separated in frequency by only

$$\Delta\nu = \frac{\nu_G}{\gamma} \sim 10^{-3} \text{ Hz} .$$

Although this is not formally a continuous spectrum, the distortions caused by even tiny fluctuations in electron energy, magnetic field strength, or pitch angle cause frequency shifts larger than  $\Delta\nu$ , so the spectrum of synchrotron radiation is effectively continuous.

Thus the synchrotron spectrum of a single electron is fairly flat at low frequencies and tapers off at frequencies above

$$\nu_{\text{max}} \approx \frac{1}{2\Delta t} \approx \pi\gamma^2\nu_G \sin\alpha .$$

It isn't really necessary to calculate the Fourier transform of the pulse shape precisely because astrophysical sources don't contain electrons with just one energy and one pitch angle in a uniform magnetic field. The actual energy distribution of cosmic rays in our Galaxy is a very broad power law, and this smears out the details of the spectrum from each energy range. Just for the record, the synchrotron power spectrum of a single electron is

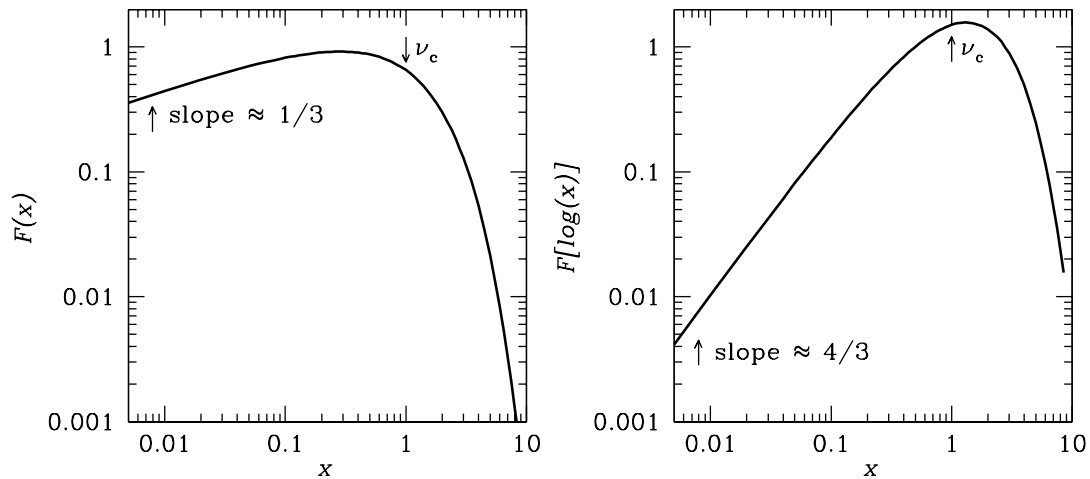
$$P(\nu) = \frac{\sqrt{3}e^3 B \sin\alpha}{mc^2} \left(\frac{\nu}{\nu_c}\right) \int_{\nu/\nu_c}^{\infty} K_{5/3}(\eta) d\eta \quad (5C1)$$

where  $K_{5/3}$  is a modified Bessel function and  $\nu_c$  is the **critical frequency** whose value is

$$\nu_c = \frac{3}{2}\gamma^2\nu_G \sin\alpha \quad (5C2)$$

(For the full mathematical derivation of this result, see Pacholczyk's *Radio Astrophysics*, a useful reference book for those who must get into the nitty gritty of radiation processes.)

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The synchrotron spectrum of a single electron in terms of

$$F(x) \equiv x \int_x^\infty K_{5/3}(\eta) d\eta,$$

where  $x \equiv \nu/\nu_c$ . Note that the spectrum is more sharply peaked than it appears in the somewhat deceptive left log-log plot because  $F(x)$  is the spectral power per unit frequency, not per unit  $\log(\text{frequency})$ . The right panel shows the power per unit  $\log(\text{frequency})$  and has a slope of  $4/3$  at low frequencies.

The spectrum of a single electron has a logarithmic slope

$$d \log P(\nu) / d \log \nu \approx \frac{1}{3}$$

at low frequencies, a broad peak near the critical frequency  $\nu_c$ , and falls off sharply at higher frequencies. One way to look at  $\nu_c$  is

$$\nu_c = \left( \frac{3}{2} \sin \alpha \right) \left( \frac{E}{mc^2} \right)^2 \frac{eB}{2\pi m_e c} \propto E^2 B_\perp.$$

That is, the frequency at which an electron emits most strongly is proportional to the square of the electron energy multiplied by the strength of the perpendicular component of the magnetic field.