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Generalized Spherical Harmonics for All-Sky Polarization Studies

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Abstract. When whole-sky linear polarization is expressed in terms of Stokes parameters T_Q and T_U, as in analyzing polarization results from the Differential Microwave Radiometers (DMR) on NASA's Cosmic Background Explorer (*COBE*), coordinate transformations produce a mixing of T_Q and T_U. Consequently, it is inappropriate to expand T_Q and T_U in ordinary spherical harmonics. The proper expansion expresses both T_Q and T_U simultaneously in terms of a particular order of generalized spherical harmonics. The approach described here has been implemented, and is being used to analyze the polarization signals from the DMR data.

1. Definition and Motivation

Generalized spherical harmonics are an extension of ordinary spherical harmonics, intended for expansion of functions whose transformation properties at each point on the sphere are more complex than just scalars. The general form

$$T^{\ell}_{n,m}(\theta,\phi) = e^{im\phi}P^{\ell}_{n,m}(\theta)$$

has three indices ℓ, m, and n where $-\ell \leq m \leq \ell$ and $-\ell \leq n \leq \ell$ (Gel'fand et al. 1963). The forms appropriate for expanding complex Stokes parameters T_Q and T_U are (Sazhin & Korolëv 1985)

$$T_Q + iT_U = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell,m}T^{\ell}_{2,m}(\theta,\phi)$$

$$T_Q - iT_U = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} E_{\ell,m}T^{\ell}_{-2,m}(\theta,\phi)$$

Since T_Q and T_U are real and $T^{\ell}_{-2,m} = T^{\ell}_{2,-m}$, the two expansions are degenerate, and we may restrict our consideration to the first form. Thus, for the DMR case we need only consider generalized spherical harmonics with $n = 2$, which we will henceforth refer to as T^{ℓ}_{m}.

The $D_{\ell,m}$ are complex expansion coefficients, analogues of the $a_{\ell,m}$ of ordinary spherical harmonic expansions of scalar quantities. Like them, the $D_{\ell,m}$ for a given ℓ transform among themselves in a coordinate transformation, but the absolute sum $\sum_m D_{\ell,m}\overline{D_{\ell,m}}$ is invariant.

Following recent work by Zaldarriaga & Seljak (1997) and Kamionkowski et al. (1997), we can partition the $4\ell+2$ independent real parameters per value of ℓ into those associated with even-parity solutions and odd-parity solutions, called E-like and B-like respectively by Zaldarriaga & Seljak. The formula appropriate for the phase convention used here is

$$\begin{aligned} D^{E}_{\ell,m} &= -(D_{\ell,m} + (-1)^{\ell+m}\overline{D_{\ell,-m}})/2 \\ D^{B}_{\ell,m} &= i(D_{\ell,m} - (-1)^{\ell+m}\overline{D_{\ell,-m}})/2 \end{aligned} \tag{1}$$

2. Properties and Computation

- Generalized spherical harmonics start at $\ell = 2$, and for each ℓ, $-\ell \leq m \leq \ell$.
- $P^{\ell}_{2,-m}(\theta) = P^{\ell}_{2,m}(180° - \theta)$.
- $P^{\ell}_{2,m}$ is real for m even, and pure imaginary for m odd.
- All functions are zero at the poles except $P^{\ell}_{2,2}$, which is nonzero at the North Pole ($\theta = 0°$), and $P^{\ell}_{2,-2}$, nonzero at the South Pole ($\theta = 180°$).
- Functions are normalized such that for any value of ℓ, the integral over the sphere of the sum of squares for all m gives unity. Thus the "strength" of an individual function decreases as ℓ increases when contrasted with the usual normalization for ordinary spherical harmonics, where each m individually integrates to unity.
- Function evaluation is by recursion. Recurrences on ℓ and then on m are used to reach each particular function. (Note that θ here refers to the colatitude, not the latitude.)

$$\begin{aligned} &\frac{\sqrt{(\ell+m+1)(\ell-m+1)(\ell+j+1)(\ell-j+1)}}{(2\ell+1)(\ell+1)} P^{\ell+1}_{j,m}(\theta) + \frac{mj}{\ell(\ell+1)} P^{\ell}_{j,m}(\theta) + \\ &\qquad \frac{\sqrt{(\ell+m)(\ell-m)(\ell+j)(\ell-j)}}{\ell(2\ell+1)} P^{\ell-1}_{j,m}(\theta) = \cos\theta\, P^{\ell}_{j,m}(\theta) \end{aligned} \tag{2}$$

$$\begin{aligned} &\sqrt{(\ell+m+1)(\ell-m)}\,P^{\ell}_{j,m+1}(\theta) - \sqrt{(\ell+m)(\ell-m+1)}\,P^{\ell}_{j,m-1}(\theta) = \\ &\qquad 2i\frac{m\cos\theta - j}{\sin\theta} P^{\ell}_{j,m}(\theta) \end{aligned} \tag{3}$$

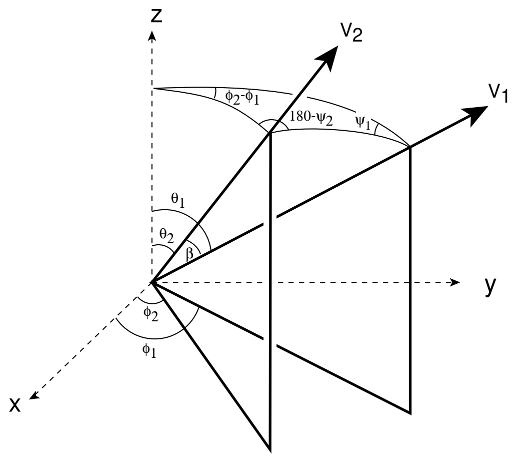

Figure 1. Geometry for definitions of ψ_1 and ψ_2 (Kosowsky 1996).

- The recursion is anchored by explicit formulas for the generalized quadrupole $P^2_{2,m}(\theta)$. (The recursion in ℓ at $\ell = 2$ defines $\ell = 3$, since the coefficient for $\ell = 1$ vanishes.)

$$
\begin{aligned}
P^2_{2,-2}(\theta) &= \frac{1}{4}(\cos\theta - 1)^2 \\
P^2_{2,-1}(\theta) &= \frac{i}{2}\sin\theta(\cos\theta - 1) \\
P^2_{2,0}(\theta) &= \sqrt{\frac{3}{8}}(\cos^2\theta - 1) \\
P^2_{2,1}(\theta) &= \frac{i}{2}\sin\theta(\cos\theta + 1) \\
P^2_{2,2}(\theta) &= \frac{1}{4}(\cos\theta + 1)^2
\end{aligned}
\tag{4}
$$

3. Sum Rules and Correlation Functions

Generalized spherical harmonics obey a sum rule analogous to a familiar one for ordinary spherical harmonics, but it includes an explicit phase factor which depends on the orientation of the two lines of sight. That phase factor depends on the angle ψ which carries the reference direction for line of sight $\vec{v}_1$ into the reference direction for line of sight $\vec{v}_2$. The geometry of $\vec{v}_1$ and $\vec{v}_2$ is illustrated in Figure 1. $\psi = \psi_1 + \psi_2$, which has the following geometric interpretation. The reference direction is rotated by ψ_1 into the great circle from $\vec{v}_1$ to $\vec{v}_2$, translated to $\vec{v}_2$, and then rotated through ψ_2 to bring it into alignment with the reference direction at $\vec{v}_2$. (β is the angle between $\vec{v}_1$ and $\vec{v}_2$).

$$
\begin{aligned}
\cos\beta &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1) \\
\sin\psi_1 &= \sin\theta_2\sin(\phi_2 - \phi_1)/\sin\beta
\end{aligned}
$$

$$\begin{aligned}\cos\psi_1 &= (\sin\theta_2\cos\theta_1\sin(\phi_2-\phi_1)-\sin\theta_1\cos\theta_2)/\sin\beta\\ \sin\psi_2 &= \sin\theta_1\sin(\phi_1-\phi_2)/\sin\beta\\ \cos\psi_2 &= (\sin\theta_1\cos\theta_2\sin(\phi_2-\phi_1)-\sin\theta_2\cos\theta_1)/\sin\beta \end{aligned} \tag{5}$$

With that definition of ψ, the sum rule relating generalized spherical harmonics along two lines of sight to the angle β between those lines of sight is

$$P^\ell_{2,2}(\beta) = e^{-2i\psi}\sum_{m=-\ell}^{\ell} T^\ell_{2,m}(\theta_1,\phi_1)\overline{T^\ell_{2,m}(\theta_2,\phi_2)}.$$

If this phase factor is included in the definition of the spherical average over all directions $\vec{v}_i$ and $\vec{v}_j$ separated by an angle β

$$C(\beta) =< T_Q(\vec{v}_i)T_Q(\vec{v}_j) + T_U(\vec{v}_i)T_U(\vec{v}_j) >= \sum_{\beta_{ij}=\beta} e^{-2i\psi} Z(\vec{v}_i)\overline{Z(\vec{v}_j)}$$

where $Z = T_Q + iT_U$, then this allows us to define rotationally invariant analogues C^P_ℓ to the power spectra C_ℓ:

$$C(\beta) = \sum_\ell C^P_\ell P^\ell_{2,2}(\beta) = \sum_\ell P^\ell_{2,2}(\beta)\sum_m D_{\ell,m}\overline{D_{\ell,m}}.$$

Additionally, we can construct analogous sums of $D^E_{\ell,m}$ and $D^B_{\ell,m}$, which we denote as C^E_ℓ and C^B_ℓ respectively. These are the appropriate quantities to use for comparison to theoretical treatments of polarization. The partitioning into C^E_ℓ and C^B_ℓ is pertinent since Zaldarriaga & Seljak (1996) and Kamionkowski et al. (1996) both show that scalar perturbations cannot produce a nonzero C^B_ℓ.

It is interesting to note that $P^l_{2,2}(\cos(180°)) = 0$, which implies that correlations between physical polarization signals vanish at the antipodes.

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