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A New Stable Method for Long-Time Integration in an N-Body Problem

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Abstract. The most serious error in numerical simulations is the accumulation of discretization error due to the finite stepsize. Traditional integrators such as Runge-Kutta methods cause linear secular errors to the energy, the semi-major axis, and the eccentricity of orbiting objects. Potter (1973) described an implicit second-order integrator for particles in a plasma with a magnetic field. We have used this integrator for an investigation of the dynamics of particles around a planet (or star) in a co-rotating coordinate system. A big advantage of this numerical integrator is its stability: the error in the semi-major axis and the eccentricity depends only on the step size and does not grow with an increasing number of time steps. The argument of pericenter changes linearly with time and more slowly than in the case of the Runge-Kutta integrator. In addition, this implicit integrator takes much less computing time than the second-order Runge-Kutta method. We tested this method for several astronomical systems and for motion of an asteroid in a 1:1 Jupiter resonance during 200 million time steps (about 5 million years or 800 thousand periods of asteroid resonance motion).

1. Introduction

In the last few years there has been great interest in the numerical study of long term evolution of bodies of the Solar system. As the integration time increases, the numerical results become more contaminated by various errors. The most serious error is the accumulation of the discretization (truncation) error due to a finite stepsize (or the replacement of continuous differential equations by finite difference equations). The conventional integrators such as Runge-Kutta, multistep and Taylor methods, generate linear secular errors in orbital energy and angular momentum. This means that the semi-major axis and the eccentricity change linearly with time and the linear secular error in the semi-major axis produces a quadratic secular error in the planetary longitude. A new symplectic integrator produces no secular truncation errors in the actions of a Hamiltonian system.

2. Implicit Integrator

In this paper we briefly discuss an implicit numerical integrator. The discretization errors in the energy, the semi-major axis, and the eccentricity by the implicit second-order integrator show only periodic changes. The truncation error in the argument of pericenter grows linearly in time. The equations of motion of a particle in the gravitational field of the Sun and the planet with mass m_{pl} in the corotating coordinate system take the form:

$$\begin{aligned} \ddot{x} &= 2\dot{y} + x + F_x \\ \ddot{y} &= -2\dot{x} + y + F_y \\ \ddot{z} &= F_z \quad , \end{aligned} \tag{1}$$

where:

$$\begin{aligned} F_x &= -\frac{x}{(x^2+y^2+z^2)^{3/2}} - \frac{m_{pl}(x-x_{pl})}{\left[(x-x_{pl})^2+(y-y_{pl})^2+(z-z_{pl})^2)\right]^{3/2}} \\ F_y &= -\frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{m_{pl}(y-y_{pl})}{\left[(x-x_{pl})^2+(y-y_{pl})^2+(z-z_{pl})^2)\right]^{3/2}} \\ F_z &= -\frac{z}{(x^2+y^2+z^2)^{3/2}} - \frac{m_{pl}\, z}{\left[(x-x_{pl})^2+(y-y_{pl})^2+(z-z_{pl})^2)\right]^{3/2}} \quad . \end{aligned}$$

Here the total mass of the Sun and the planet is taken as the unit of mass, and the distance between the planet and the Sun is taken as the unit of length. The unit of time is chosen in such a way that the angular velocity of orbital motion of the planet is equal to unity, and, hence, its orbital period is 2π. Let $v^* = v_x + i\, v_y$, $x^* = x + i\, y$, and $F^* = F_x + i\, F_y$. We obtain rather than (1):

$$\begin{aligned} \frac{dz}{dt} &= v_z \\ \frac{dv_z}{dt} &= F_z \\ \frac{dx^*}{dt} &= v^* \\ \frac{dv^*}{dt} &= -2\, i\, v^* + x^* + F^* \quad . \end{aligned} \tag{2}$$

We may solve the equation $\frac{dU(t)}{dt} + R(U(t),t) = 0$ with initial conditions $U(t_0) = U_0$ by the implicit second-order integrator described in Potter (1973):

$$v^{[n+1]} = v^{[n]} - \frac{1}{2}\,(R^{[n]} + R^{[n+1]})\,\Delta t \quad . \tag{3}$$

In our equations (2), R is a function of x, y, z. We will calculate this function R in space-time points $n + 1/2$. From (2) by the use of Eq. (3) we derive the equations for new integrator (Taidakova 1990, 1995):

$$v_x^{[n+1]} = \frac{v_x^{[n]}(1-\Delta^2 t) + (2v_y^{[n]} + x^{[n+\frac{1}{2}]} + F_x^{[n+\frac{1}{2}]})\Delta t + (y^{[n+\frac{1}{2}]} + F_y^{[n+\frac{1}{2}]})\Delta^2 t}{1+\Delta^2 t}$$

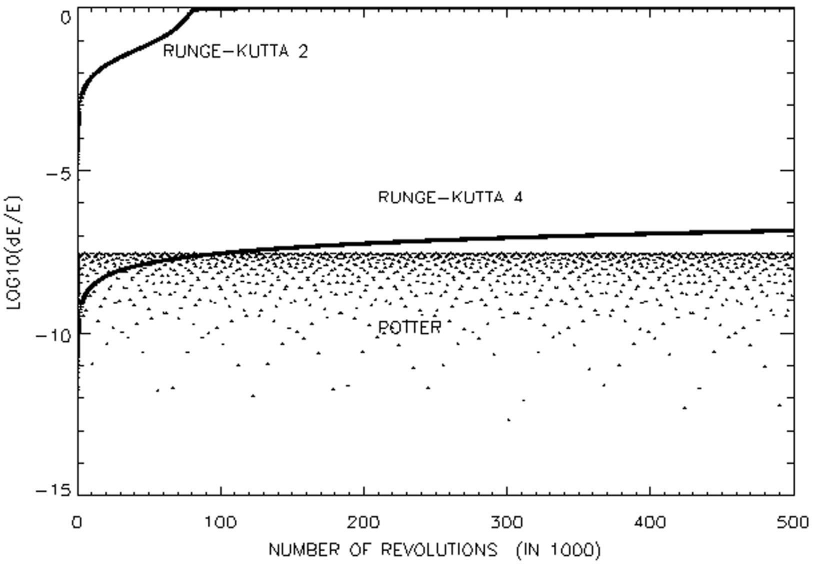

Figure 1. Numerical errors in the energy with second order Runge-Kutta, fourth order Runge-Kutta and second order Potter integrator.

$$
\begin{aligned}
v_y^{[n+1]} &= \frac{v_y^{[n]}(1-\Delta^2 t) - (2v_x^{[n]} - y^{[n+\frac{1}{2}]} - F_y^{[n+\frac{1}{2}]})\Delta t + (x^{[n+\frac{1}{2}]} + F_x^{[n+\frac{1}{2}]})\Delta^2 t}{1+\Delta^2 t} \\
v_z^{[n+1]} &= v_z^{[n]} + F_z^{[n+\frac{1}{2}]}\Delta t \\
x^{[n+1]} &= x^{[n]} + (v_x^{[n+1]} + v_x^{[n]})\Delta t/2 \\
y^{[n+1]} &= y^{[n]} + (v_y^{[n+1]} + v_y^{[n]})\Delta t/2 \\
z^{[n+1]} &= z^{[n]} + (v_z^{[n+1]} + v_z^{[n]})\Delta t/2 \quad , \qquad (4)
\end{aligned}
$$

where: $x^{[n+\frac{1}{2}]} = x^{[n]} + v_x^{[n]}\Delta t/2$; $y^{[n+\frac{1}{2}]} = y^{[n]} + v_y^{[n]}\Delta t/2$;
$z^{[n+\frac{1}{2}]} = z^{[n]} + v_z^{[n]}\Delta t/2$; $F_{x,\,y,\,z}^{[n+\frac{1}{2}]} = F\left(x^{[n+\frac{1}{2}]}, y^{[n+\frac{1}{2}]}, z^{[n+\frac{1}{2}]}, t^{[n+\frac{1}{2}]}\right)$.

3. Numerical Examples

We tested this method for several astronomical systems (Taidakova 1995). In order to see the properties of the implicit integrator, we first choose the 2-body problem. Figure 1 shows the numerical errors in the energy $\log(\Delta E/E)$ in the numerical integrations of the motion of the particle in circular orbit around the Sun with different integrators. Figure 2 shows the errors in the parameter $\Delta_i = \sqrt{(x_0 - x_i)^2 + (v_{x_0} - v_{x_i})^2 + (v_{y_0} - v_{y_i})^2}$, where i is the number of revolutions, x is the coordinate ($y = 0$), and v_x, v_y are the velocities in the numerical integration of the orbital motion of asteroid in the 1:1 Jupiter resonance during 200 million steps. The computer time with the implicit second-order integrator is about 1.52–1.94 times faster than that with second-order Runge-Kutta integrator.

The calculation were carried out in a PC with DX4/100.

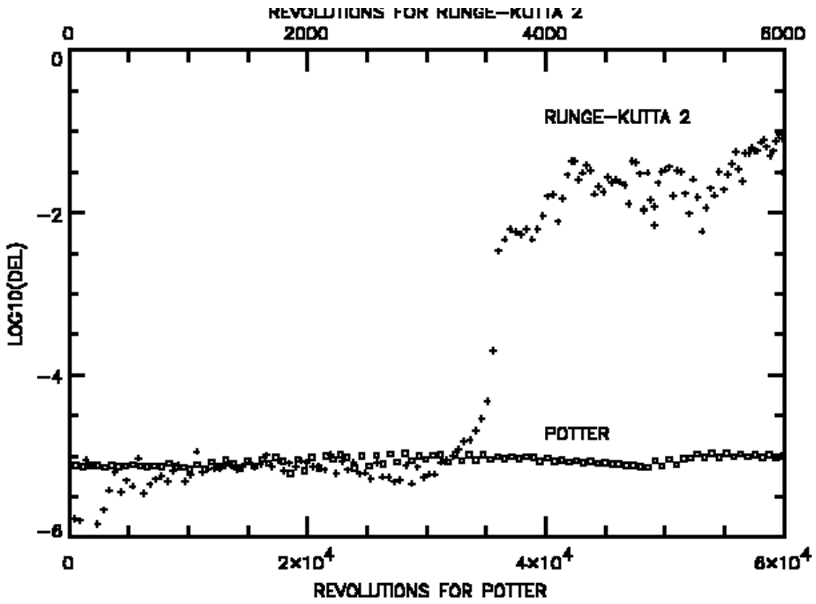

Figure 2. Numerical errors in the parameter Δ_i with second order Runge-Kutta method and second order Potter integrator.

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