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The Time Interferometer: Synthesis of the Correlation Function

V. V. Vityazev

Astronomy Department, St. Petersburg University, Bibliotechnaya pl.2, Petrodvorets, St. Petersburg, 198904, Russia.

Abstract. A comparative study of the fundamentals, problems and techniques common to the spectral analysis of time series and interferometry is presented. The aperture synthesis technique well known in radio astronomy is adapted to the spectral analysis of the time series.

1. Review of the Fundamentals

Suppose that a time series $X(t)$ is a stationary stochastic process with zero mean defined by a set of realizations

$$X(t) = \{x_p\}_{p=1}^{N}, \quad 0 \leq t \leq T < \infty. \tag{1}$$

To describe gaps in observations we introduce the *time window function*

$$h(t) \;=\begin{cases} 1 & \text{if at time } t \text{ the data exists} \\ 0 & \text{if at time } t \text{ the data is absent.} \end{cases} \tag{2}$$

With this notation the observed time series can be represented as

$$Y(t) = h(t)X(t). \tag{3}$$

Calculate the *periodogram*:

$$D(\omega) = \frac{1}{2\pi T} < \left| \int_0^T y_p(t) \exp(-i\omega t) dt \right|^2 >, \tag{4}$$

where <> denotes averaging over the set of realizations. Under the conditions stated above, the relationship between the periodogram $D(\omega)$ and the power spectrum $G(\omega)$ is given by the convolution

$$D(\omega) = \int_{-\infty}^{+\infty} W(\omega')G(\omega - \omega')d\omega', \tag{5}$$

where the *spectral window function* $W(\omega)$ is the periodogram of the time window function

$$W(\omega) = \frac{1}{2\pi T} \left| \int_0^T h(t) \exp(-i\omega t) dt \right|^2. \tag{6}$$

Now, from Eq. (5), for the correlogram $k_D(\tau)$ and the correlation window $H(\tau)$, introduced as inverse Fourier transforms of $D(\omega)$ and $W(\omega)$ respectively, one has:

$$k_D(\tau) = H(\tau)k(\tau), \tag{7}$$

where $k(\tau)$ is the auto-correlation function of $X(t)$. For further details the reader is referred to Jenkins & Watts (1968), Deeming (1975), Otnes & Enocson (1978), Marple (1978), and Terebish (1992).

In interferometry (Esepkina et al. 1973; Thompson et al. 1986), the *intensity of radiation* from any source on the sky can be described in terms of position (α, δ), wavelength (λ), and time (t). Each measurement is an average over a band of wavelengths and over some span of time. Here, the one-dimensional, monochromatic, and instantaneous approximation is used to simplify the discussion. The resulting specific intensity $T_b(\vartheta)$ that describes the distribution of the source brightness along the arc of a circle (ϑ is the angular coordinate) has a Fourier transform $\hat{T}_b(u)$, which is called *the spectrum of spatial frequencies*. Correspondingly, when an interferometer measures *the visibility data* $\hat{T}_a(u)$, the image, or *the map*, $T_a(\vartheta)$ can be calculated by the Fourier transform of $\hat{T}_a(u)$. Two fundamental relations are valid:

$$T_a(\vartheta) = A(\vartheta) \otimes T(\vartheta), \tag{8}$$

$$\hat{T}_a(u) = \hat{A}(u) \times \hat{T}(u), \tag{9}$$

where $A(\vartheta)$ is *the beam* of interferometer and $\hat{A}(u)$ is *the transfer function*. Eq. (9) determines an interferometer as a *filter of spatial frequencies*, whereas Eq. (8) explains why, due to convolution of $T_b(\vartheta)$ with the beam $A(\vartheta)$, the resulting image is called a *dirty map*.

Upon close examination one can see that in both sciences the rigorous (theoretical) quantities are introduced at the first level. In the spectral-analysis case they are the *power spectrum* and the *correlation function*; in the interferometry their counterparts are the *distribution of brightness* and the *spatial spectrum*. At the second level we have estimators of the strict quantities. In spectral analysis, they are the *periodogram* and the *correlogram*, whereas in the interferometry these are the *map* and the *visibility data*, respectively. Finally, equations which connect the quantities of the two levels are identical (convolution and multiplication) and include the characteristics of observations: the *beam* and the *transfer function* and their analogs, i.e., the *spectral window* and the *correlation window*.

In reality, due to the finite dimensions of mirrors and the finite time spans of observations we cannot get the true quantities, and all we can do is to find their optimal approximations. In optics or in radio astronomy, when filled apertures are used, the maps are produced directly in the focal plane of a telescope. Analogously, when the time series is given at all points of some interval or at time points regularly spaced within the interval, the evaluation of the periodogram can be made quite easily. When an interferometer is used, the aperture is not solid, and what we can measure is the visibility data, i.e., the estimator of the spectrum of spatial frequencies. The longer the baseline, the smaller the area of the $(u - v)$-plane (u-domain in the one-dimensional case) filled, and the more dirty the resulting map becomes.

To overcome this, various techniques of *aperture synthesis* are used, and this leads to complete solution of the problem since the $(u - v)$-plane is completely filled. If the aperture synthesis provides partial filling of the $(u - v)$-plane, the *cleaning* procedures can be used with the aim of eliminating the artifacts of the "holes" in the $(u - v)$-plane from the map. We have the same problems in the spectral-analysis case, when the time points are distributed irregularly or have long gaps. In this case the correlograms cannot be determined for all values of time lag τ, and this would give false features in the resulting periodograms. The main aim of the present paper is to answer the question: is it possible to apply the aperture synthesis method to spectral analysis of time series?

2. Synthesis of the Correlation Function

It is known that to do aperture synthesis one must have an interferometer with the changeable baseline. Of all the schemes of the aperture synthesis the one proposed by Ryle (1960) is the most suitable for us. This consists of two antennas A and B fixed at separation L. The third antenna C is moving inside the interval $[L/2, L]$. At each position of the moving antenna one obtains two interferometers (CB) and (AC) with the baselines l and $L/2 + l$, and they yield the values of the visibility data $\hat{T}_a$ at the points $u = l/\lambda$ and $u = U/2 + l/\lambda$, where $U = L/\lambda$. Obviously, while antenna C sweeps all the interval $[L/2, L]$, the visibility data $\hat{T}_a(u)$ become available at all points of the interval $[0, U]$.

Now we apply this idea to time series analysis. Every two points spaced at the distance τ may be called *the Time Interferometer* with variable baseline τ, since each such pair yields the estimation of the correlation function $k_s(\tau)$ by averaging the products $X(t)X(t + \tau)$ over the set of realizations. The value $k_s(\tau)$ can be obtained no matter where the points t and $t + \tau$ are located inside the interval $[0, T]$. This follows from our assumption that $X(t)$ is a stationary stochastic process, and that is crucial for our study.

Assume that the time series is given at two points t_1 and $t_1 + T/2$. This Time Interferometer allows us to get the correlation function only at the point $\tau = T/2$. Let us make new observations at the points $t_1 + T/2 < t < t_1 +$ T. It is clear that each new point $t = t_1 + T/2 + \tau$ yields two additional values of the correlation function, namely, at the points τ and $T/2 + \tau$. Obviously, when the observations cover the interval $[t_1 + T/2, t_1 + T]$, the values of $k_s(\tau)$ become available at all points of the interval $[0, T]$.

Now we see that the fixed antennas A and B in the Ryle interferometer are the counterparts of the boundary points t_1 and $t_1 + T$, while each new position of the moving antenna C is nothing else but the new point of observations. Since Ryle's interferometer makes the synthesis of the visibility data, it is a good reason to call the described procedure the *synthesis of the correlation function.*

Obviously, complete synthesis and the evaluation of $k_s(\tau)$ from an even time series are the same. Thus, with $h(t) \equiv 1$ elsewhere for the synthesized periodogram we get

$$D_s(\omega) = \frac{1}{\pi} \int_0^T (1 - \frac{\tau}{T}) k_s(\tau) \cos(\omega\tau) d\tau. \tag{10}$$

This estimator yields the clean spectrum, i.e., free of the false peaks that come from the "holes" in the τ-domain.

To proceed further, assume that we have a gap in the observations. Let the length of the gap be l and the longest distance between the borders of the gap and the boundary points of the interval $[0, T]$ be a. If $l \leq a$ then the correlation function can be evaluated at all points $\tau \in [0, T]$, otherwise only on the subintervals $[0, a]$ and $[l, T]$. In this case the correlation function turns out to be synthesized at all points except the "hole" of the length $l - a$, and, consequently, the synthesized periodogram $D_s(\omega)$ calculated from Eq. (10) will not be clean. Nevertheless, it is less contaminated than the periodogram calculated directly from Eqs. (4). Thus we see that to clean the spectrum completely, one needs to perform more observations until the condition $l \leq a$ becomes true.

When the only realization is available, averaging over the ensemble should be replaced with averaging over time. For further development of the method the reader is referred to Vityazev (1996).

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