On–axis Instrumental Polarization Calibration for Linear Feeds

W. D. Cotton, August 7, 2015

Abstract—Various instrumental and atmospheric effects corrupt the response of an interferometer to polarized signals. In the case of high dynamic range imaging, uncorrected, these effects can also degrade the total intensity image. These effects must be estimated and removed in order to produce images of the polarized emission, or high dynamic range total intensity images. This memo describes the implementation in Obit of the feed ellipticity–orientation modeling and correction for arrays with linearly polarized feeds. Examples are given using data from the KAT-7 and ALMA arrays.

Index Terms—interferometry, polarization, calibration

I. INTRODUCTION

Radio interferometric imaging of polarized celestial emission provides a powerful probe of various physical processes as well as the propagation through intervening media. Instrumental and atmospheric effects corrupt the response of an interferometer to polarized signals. In particular, the array detectors do not respond to precisely the intended response of an interferometer to polarized signals. In particular, arrays used systems are right- and left-hand circular polarizations. In practice, the detected polarizations are never precisely the desired ones. The exact polarization states detected must be determined in order to transform the measured visibilities into the corrected form.

This memo describes an implementation in the Obit package [1] of the nonlinear ellipticity–orientation model for arrays using alt-az mounted antennas and using detectors (AKA “feeds”) sensitive to orthogonal linear polarizations. The result of the application of the calibration derived is the observed data corrected and transformed into a circular basis. Calibration of arrays with circular feeds is discussed in [2]. The development here follows that of [3].

II. INTERFEROMETRIC POLARIMETRY

An arbitrary electromagnetic wave can be described as elliptically polarized, circular and linear polarization are extreme cases. One way of modeling the polarization state to which a detector responds is the ellipticity and orientation of the ellipse to which the detector responds. An alternate, and popular approach is to model the response as the desired state plus a fraction of the orthogonal state. The fraction of the orthogonal state is referred to as the “leakage term”. This model has the advantage that it can be linearized allowing for faster fitting. See [4][5][6] for more detailed descriptions of the response to polarized radiation.

Much traditional radio interferometric software at the NRAO uses data from arrays using circular feeds. Therefore, it is convenient to have the transformation of observed to calibrated data to also transform the visibilities into a circular basis, i.e. what would have been observed with circular feeds.

A. Effects of prior calibration

Discussions of instrumental polarization such as those given in [4] generally do not include the effects of prior calibration on the data. The common practice is to calibrate the two parallel polarization systems of visibilities independently.

Since interferometers only measure differences of the phase of a wavefront between pairs of antennas, the absolute phase is undetermined and phases are referred to that at a reference antenna whose phase (and delay) has been arbitrarily set to zero. This allows for an arbitrary phase and delay offset between the independent polarization systems. See [7] for a discussion of correcting these offsets.

B. Response by Linear Feeds

For an interferometer with perfectly linear feeds observing an unresolved, partially polarized source and forming all four correlation products the expected visibility \[[X,X,XY,YX,YY]\] is given by [8]:

\[
S_L = [ipol + qpol \cos(2\psi) + upol \sin(2\psi),
- qpol \sin(2\psi) + upol \cos(2\psi) + j \upsilon_{pol},
- qpol \sin(2\psi) + upol \cos(2\psi) - j \upsilon_{pol},
ipol - qpol \cos(2\psi) - upol \sin(2\psi)]
\]

where \(ipol, qpol, upol\) and \(\upsilon_{pol}\) are the Stokes’ \(I, Q, U\) and \(V\) of the source and \(j\) is \(\sqrt{-1}\). \(\psi\) is the angle between the “\(X\)” feed and the meridian measured from north towards east. This is a combination of the orientation of the feed wrt the antenna \(\phi\) and the orientation of the antenna wrt the sky \(\chi\). The “\(Y\)” is then presumed to be oriented by a further \(90^\circ\) from the \(X\) feed. \(\chi\) is the parallactic angle given by:

\[
\chi = \tan^{-1}\left(\frac{\cos \lambda \sin h}{\sin \lambda \cos \delta - \cos \lambda \sin \delta \cos h}\right)
\]  \((1)\)
where $\delta$ is the source declination, $\lambda$ is the latitude of the antenna and $h$ is the hour angle of the source. For interferometers using linear feeds, Stokes Q and U contribute to all four correlation products.

For an interferometer using perfect circular feeds, the equivalent visibility $[RR,RL,LR,LL]$ is:

$$S_C = [ipol + vpol, qpol + j upol, qpol - j upol, ipol - vpol]$$

(2)

In this case, the cross-polarized components (RL, LR) give the linearly polarized response to the source without a dependence on the parallactic angle. In practice, the feeds are never linearly polarized response to the source without a dependence on the parallactic angle. To separate the two contributions to the polarized response requires that at least one calibrator be observed over a sufficient range of parallactic angle to separate the two effects. How much is enough depends on the SNR but generally a radian or more is desirable. For a calibrator of known polarization, including none, any distribution of parallactic angle is usable. A calibrator of known polarization angle is required to constrain the feed orientation.

D. Absolute v. Relative Calibration

To first order, the equations describing the polarized response for a short baseline interferometer are degenerate in that there are more parameters than independent measurements. This degeneracy is broken by the higher order terms if they are large enough. In practice, feeds with linear polarization frequently have low polarization imperfections and other constraints may be needed. These may be that a given (reference) antenna is defined to be “perfect” and its parameters fixed or that an average over the array of some terms if they are large enough. In practice, feeds with linear polarization frequently have low polarization imperfections and other constraints may be needed. These may be that a given (reference) antenna is defined to be “perfect” and its parameters fixed or that an average over the array of some parameter is “perfect”.

III. SOLUTIONS FOR MODEL PARAMETERS

The parameters of the model described in Section II-B can be fitted to a data set using any of a number of techniques. In the following two such techniques are explored.

A. Relaxation fitting

Following the method of [2] the first technique considered is a relaxation technique in which corrections are cyclically made to each model parameter being fitted by determining the effect of that parameter on the $\chi^2$. The $\chi^2$ of the fit is defined as

$$\chi^2 = \sum_{i=0}^{n} (model_i - obs_i)^2 / \sigma_i^2$$

where $model_i$ is the model value for observation $i$ (real or imaginary part of visibility), $obs_i$ is the observation for $i$ and $\sigma_i^2$ is the variance of the amplitude of observation $i$. The model values are components of the vector $V_{model}$ given by Eq. 3. For each parameter, $P$, in each iteration, $n$, the revised value is given by:

$$P_{n+1} = P_n + \tan^{-1} \left( \frac{\partial \chi^2}{\partial P} \right)$$

subject to the constrain that the corrected value, $P_{n+1}$, leads to a decrease in $\chi^2$. Should this not be the case, the correction is reduced in magnitude until the $\chi^2$ decreases. Note: the evaluation of $\chi^2$ only need consider data which depend on the given parameter. The various partial derivatives are given in the appendix. In the following, this is referred to as the “fast” technique.
B. General nonlinear least squares

Most scientific software packages provide a generalized nonlinear least squares facility such as the Levenberg-Marquardt least squares technique. These can provide robust solutions as well as an error analysis but can be expensive to compute. An initial estimate of the parameters using the relaxation technique can reduce the total computing cost. These generalised nonlinear least squares packages generally need the derivatives of either the \( \chi^2 \) or the model wrt the various parameters; these derivatives are given in the appendix.

IV. Correcting Observed Visibilities

Once the model parameters describing the instrumental polarization are known, they can be used to remove the instrumental effects from the data. The corrected visibility vector \( V'_{\text{cor}} \) can be obtained from the observed visibility vector \( V_{\text{obs}} \) by

\[
V'_{\text{cor}} = M^{-1} V_{\text{obs}}
\]

where \( M^{-1} \) is the inverse of the Muller matrix. A useful property of the outer product is that the the outer product of the inverses of two Jones matrices is the inverse of the outer product of the Jones matrices themselves. The inverse of a \( 2 \times 2 \) matrix given by:

\[
J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

is

\[
J^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

In the implementation described here, the multiplication by the inverse Muller matrix will transform the observed [XX,XY,YX,YY] visibilities into corrected [RR,RL,LR,LL] visibilities. These then need correction for the parallactic angle:

\[
V'_{\text{cor}} = \begin{bmatrix} \Delta \chi_{ik} & 0 & 0 & 0 \\ 0 & \Sigma \chi_{ik} & 0 & 0 \\ 0 & 0 & \Sigma \chi_{ik}^* & 0 \\ 0 & 0 & 0 & \Delta \chi_{ik}^* \end{bmatrix} V_{\text{cor}}
\]

A. Y/X Gain

The parallel hand calibration is generally done assuming unpolarized calibrators. As was shown in Section II-B, actual calibrator polarization will have an additive effect of the parallel hand visibilities with opposite signs and which vary with parallactic angle. If an unpolarized source is included in the data, it can be incorporated in the the parallel hand calibration to fix the relative gains in the calibration process and then constrain further calibrations to average the parallel hand calibration to average the parallel hand data before calibration. This will leave an error in the Y/X gain ratio determined by the calibrator polarization and the parallactic angle of the calibration. Solving for the Y/X gain ratio can be included in the calibration fitting.

V. Obit Implementation: PCal

Implementation of this calibration technique in Obit is in the task PCal which uses the software class ObitPolnCalFit. Multiple calibrators may be included in the solution. Calibration is per channel or sliding window of channels. Fitted feed parameters are stored in an AIPS PD table with the “Real” part being the ellipticity and the “Imaginary” being orientation. The table is labeled as type “ORI-ELP”.

Fitting always uses the “fast” Relaxation method optionally followed by a Levenberg-Marquardt least squares using the GSL package nonlinear fitter. The initial value of the parameters are for perfect feeds for the first channel fitted and the results of the previous channel in subsequent fittings.

The parameters to PCal allow specifying the source polarizations for some subset of the calibrators in terms of the fractional polarization, the EVPAx2 and the rotation measure. In this case, the X-Y phase offset in the data should also be solved for and the results stored in the AIPS PD table as well as an AIPS BP table.

Application of instrumental polarization corrections to the data computes the inverse Muller matrix from the outer product of the inverse of the Jones matrices. The antenna Jones matrices are derived as described in Section II-B.

A. Feed orientations

In order to constrain the orientations of feeds, observations of a source of known polarization angle is needed. Fortunately, for feeds sensitive to linear polarization the orientation of the feed can be readily measured. If the feed orientation are known to sufficient accuracy (and given in the Antenna table) they need not be fitted. Obit/PCal allow either fitting for the feed orientation or leaving it at the nominal value.

B. X-Y phase difference

Typically, the calibration of the parallel hand gains is done independently leaving an arbitrary difference in phase and group delay between the two systems. An initial estimate of the delay difference can be obtained and removed using Obit task RLDly [7]. Residual X-Y phase differences can be estimated in the fitting if one or more of the calibrators is polarized although its state need not be previously known. These phase differences are included in the output bandpass (AIPS BP) table.

C. Y/X Gain

As described in Section IV-A the Y/X gain ratio may not be properly determined by the parallel hand calibration and can be included in the calibration fitting. In PCal the relative gains can be determined in each channel or block of channels fitted and can be incorporated into the output bandpass table.

D. Stokes V

The value of Stokes’ V can be left at 0 or fitted on a calibrator by calibrator basis.
E. Absolute or Relative Calibration

There are several options for relative v. absolute calibration and these are controlled by parameter refAnt. If refAnt is specified as zero, then no additional constraints are imposed and an “absolute” calibration is attempted. If refAnt is greater than 0, then it indicates an antenna for which the XPol ellipticity is fixed to zero (perfect feed). If refAnt is -1 then the average of all ellipticities is forced to zero. The latter option seems most effective when Stokes V is desired.

VI. EXAMPLES WITH KAT-7 DATA

The implemention in Obit was tested using data from the KAT-7 (MeerKAT prototype) array in South Africa. This array used linear feeds oriented vertically and horizontally and operates in L Band. Data were calibrated using the strong, very weakly polarized source B1934-638 and other calibrators were self calibrated averaging the parallel hand data to avoid introducing artifacts from the calibrator’s polarization. All data sets included B1934-638 and 3C286 and the latter was used in RLDly to correct the cross polarized delay. Instrumental polarization calibration used fixed polarization models for B1934-638 (zero linear polarization) and 3C286 (0.095 fractional linear polarization at 1.3 GHz and 0.098 at 1.8 GHz at EVPA=33°). After calibration, the data were imaged and as all sources were unresolved, the value in the Stokes I, Q, U and V images at the location of the calibrator were used for evaluation. Polarization results are given in Table I and the associated image’s RMSes in Table II.

Two datasets included both 3C138 and 3C286 as polarization calibrators and were identically scheduled which allows comparisons of repeatability. These were observed at 1.3 GHz allowing the possibility of variable ionospheric Faraday rotation. The results are summarized in Tables I and II as days 1 and 2.

A third dataset included a deep integration of a polarized quasar. This dataset was calibrated and imaged as were the previous tests and the images in various Stokes parameters are shown in Figure 1 and statistical measures in Tables I and II.

VII. EXAMPLE WITH ALMA DATA

ALMA polarization commissioning data was used to derive polarized images of the well known calibrator 3C286 at 230 GHz. Data consisted of four 2 GHz bands with all correlation products recorded. The observations included calibrators 3C279, J1337-1257, and J131029+322051 for bandpass, polarization and phase calibration. ALMA uses linearly polarized feeds at 45° to either side of vertical. The calibration proceeded as follows:

1) 3C279 and J1337-1257 were used for the parallel hand group delay calibration.
2) The calibration was based on a “Flux density” of 3C279 of 10 Jy and used 3C279 as bandpass calibrator.
3) After an initial calibration based on 3C279, the calibrators and 3C286 were self calibrated and the models used in subsequent calibration (although all were unresolved in this dataset).
4) A single scan on 3C279 was used to fix the Y/X gains and subsequent calibration averaged the parallel hands before determining calibration.
5) The calibrators and 3C286 were then used to determine short period phase fluctuations,
6) followed by scan averaged amplitude and phase corrections.
7) A single scan of 3C279 was used to correct the X-Y delays.
8) Data had calibration applied using Obit task Splat.
9) Polarization calibration used 3C279, J1337-1257, and J131029+322051 solving for the source polarization, Y/X gains and X-Y phases. Since only the target (3C286) had a known polarization angle the feed orientation were left at the nominal values. Solutions were performed in blocks of 250 MHz.
10) Y data from antenna 22 were found to be unreliable and were excluded from further analysis.

After calibration, the data were imaged using Obit task Imager using short period phase self calibration followed by scan averaged amplitude and phase self calibration. As all sources were unresolved, the value in the Stokes I, Q, U and V images at the location of the calibrator were used for evaluating the polarization. Polarization results are given in Table I and the associated image’s RMSes in Table II.

<table>
<thead>
<tr>
<th>Source</th>
<th>Day</th>
<th>I rms mJy/bm</th>
<th>Q rms mJy/bm</th>
<th>U rms mJy/bm</th>
<th>V rms mJy/bm</th>
</tr>
</thead>
<tbody>
<tr>
<td>3C286</td>
<td>1</td>
<td>33</td>
<td>5.8</td>
<td>3.0</td>
<td>2.2</td>
</tr>
<tr>
<td>3C138</td>
<td>1</td>
<td>27</td>
<td>8.6</td>
<td>13.4</td>
<td>2.5</td>
</tr>
<tr>
<td>B1934-638</td>
<td>1</td>
<td>0.8</td>
<td>0.8</td>
<td>0.9</td>
<td>0.6</td>
</tr>
<tr>
<td>ph cal</td>
<td>1</td>
<td>4.7</td>
<td>0.7</td>
<td>0.9</td>
<td>0.3</td>
</tr>
<tr>
<td>3C286</td>
<td>2</td>
<td>23</td>
<td>2.3</td>
<td>2.3</td>
<td>0.2</td>
</tr>
<tr>
<td>3C138</td>
<td>2</td>
<td>26</td>
<td>8.3</td>
<td>12.0</td>
<td>3.7</td>
</tr>
<tr>
<td>B1934-638</td>
<td>2</td>
<td>3.7</td>
<td>1.0</td>
<td>1.2</td>
<td>0.7</td>
</tr>
<tr>
<td>ph cal</td>
<td>2</td>
<td>5.0</td>
<td>0.8</td>
<td>1.1</td>
<td>0.4</td>
</tr>
<tr>
<td>3C286</td>
<td>3</td>
<td>29</td>
<td>7.0</td>
<td>6.4</td>
<td>1.7</td>
</tr>
<tr>
<td>B1934-638</td>
<td>3</td>
<td>11</td>
<td>2.3</td>
<td>2.0</td>
<td>1.2</td>
</tr>
<tr>
<td>quasar</td>
<td>3</td>
<td>4.5</td>
<td>0.8</td>
<td>0.6</td>
<td>0.2</td>
</tr>
</tbody>
</table>
Fig. 1. Negative grayscale images of a quasar observed with KAT-7 in Stokes I (top left), V (top, right), Q (bottom left) and U (bottom right). Intensity scale is given by the bar at the top labeled in mJy/beam. Maximum I is 2.71 Jy/beam, Q is -27 mJy/beam, U is 81 mJy/beam. CLEAN restoring beam is shown in the lower left.
TABLE III
Calibrator polarization

<table>
<thead>
<tr>
<th>Source</th>
<th>Jy/bm</th>
<th>frac. lin. pol</th>
<th>EVPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>3C279</td>
<td>10.0</td>
<td>0.118</td>
<td>33.7</td>
</tr>
<tr>
<td>3C286</td>
<td>0.274</td>
<td>0.163</td>
<td>38.1</td>
</tr>
<tr>
<td>J1337-125</td>
<td>3.00</td>
<td>0.033</td>
<td>62.8</td>
</tr>
<tr>
<td>J131029+322051</td>
<td>0.636</td>
<td>0.048</td>
<td>85.8</td>
</tr>
</tbody>
</table>

and V images at the location of the calibrator were used for evaluation. Polarization results are given in Table I and the associated image’s RMSes in Table II.

The images of 3C286 in various Stokes parameters are shown in Figure 2 and statistical measures in Tables III and IV. Images in Stokes I, Q and U show no significant artifacts while the response in Stokes V is clearly spurious, the source position is centered between the positive and negative responses and all calibrators show the same pattern at ±0.4% of the Stokes I peak.

VIII. DISCUSSION

A method of calibrating the polarized response from radio interferometer arrays using linearly polarized feeds is described and an implementation tested using data from the KAT-7 telescope. The results in the previous section show good repeatability of results for a number of sources and a low level of residual artifacts in the derived polarization images. The low level of polarized residuals in images of the very weakly polarized source B1934-638, less than 0.1%, are indicative of the quality of correction. Polarized emission is even visible from a weaker source in the field shown in Figure 1.

A calibrator of known EVPA is needed to accurately determine the orientations of the feeds in order to ensure accurate measures of the angle of linear polarization (EVPA). This was done in the cases of the KAT-7 data tests described above. However, after calibration there were residual errors in EVPA. Polarization models are given in [9] for 3C138 (7.5% EVPA=−11°) and 3C286 (9.5% , EVPA=33°) at 1.45 GHz. 3C286 was used to set the EVPA scale but the results shown in Table I all gave an EVLA of ∼30°, 3° lower than the value used in the calibration of ∼33°. The EVPA values for 3C138 given in Table I are about 10° lower than the value given by [9]. The fractional linear polarizations given in Table I are in good agreement with [9]. The reasons for the discrepancy in EVPA, and in particular, the differences for 3C138 and 3C286 are unclear.

The ALMA example analysis gave a fractional polarization of 16.3% and an EVPA of 38.1°. This is in reasonable agreement with other measurements [10] and shows the efficacy of using the nominal ALMA feed orientations. The reasons for the clearly spurious Stokes V image of 3C286 is not understood.

TABLE IV
Image RMS

<table>
<thead>
<tr>
<th>Source</th>
<th>I rms mJy/bm</th>
<th>Q rms mJy/bm</th>
<th>U rms mJy/bm</th>
<th>V rms mJy/bm</th>
</tr>
</thead>
<tbody>
<tr>
<td>3C279</td>
<td>267</td>
<td>12</td>
<td>29</td>
<td>2.0</td>
</tr>
<tr>
<td>3C286</td>
<td>8.4</td>
<td>0.1</td>
<td>1.4</td>
<td>0.1</td>
</tr>
<tr>
<td>J1337-125</td>
<td>76</td>
<td>0.3</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>J131029+322051</td>
<td>20</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

ACKNOWLEDGMENT

I thank Rick Perley for providing his analysis of radio interferometric polarimetry. I would also like to thank Tom Maunch and Nadeem Oozeer of SKA/SA for providing KAT-7 test data and to Paulo Cortes for providing the ALMA test dataset.
Fig. 2. Negative grayscale images of 3C286 observed by ALMA in Stokes I (top left), V (top right), Q (bottom left) and U (bottom right). Intensity scale is given by the bar at the top labeled in mJy/beam. CLEAN restoring beam is shown in the lower left.
APPENDIX

Muller matrix

A polarization Jones matrix which includes the effects of parallel–hand calibration can be constructed for each antenna:

\[ \mathbf{J} = \begin{bmatrix} X,i & Y,i \end{bmatrix} = \begin{bmatrix} g_X \cos(\frac{\pi}{4} + \theta_X) e^{-j\phi_X,i} & g_X \sin(\frac{\pi}{4} + \theta_X) e^{j\phi_X,i} \end{bmatrix} \begin{bmatrix} g_Y \cos(\frac{\pi}{4} - \theta_Y) e^{j\phi_Y} & g_Y \sin(\frac{\pi}{4} - \theta_Y) e^{-j\phi_Y} \end{bmatrix} \]

where \( \theta \) is the ellipticity, \( \phi \) the orientation, \( j \) is \( \sqrt{-1} \), \( g_X \) and \( g_Y \) are corrections to the gains of the X and Y feeds resulting from the parallel hand calibration, The Muller matrix, \( \mathbf{M}_{ik} \), for baseline \( i-k \) is then the outer product of \( \mathbf{J}_i \) and \( \mathbf{J}_k^* \).

\[ \mathbf{M}_{ik} = \mathbf{J}_i \otimes \mathbf{J}_k^* \]

In order to efficiently compute the Muller matrices, for each antenna \( i \), define:

\[ C_{X,i} = \cos(\frac{\pi}{4} + \theta_{X,i}) \]
\[ C_{Y,i} = \cos(\frac{\pi}{4} - \theta_{Y,i}) \]
\[ S_{X,i} = \sin(\frac{\pi}{4} + \theta_{X,i}) \]
\[ S_{Y,i} = \sin(\frac{\pi}{4} - \theta_{Y,i}) \]

and for each baseline \( i-k \) define:

\[ \Delta \chi_{ik} = e^{j(X_i - X_k)} \]
\[ \Sigma \chi_{ik} = e^{j(X_i + X_k)} \]

The elements of the Muller matrix to convert the circular basis model visibility to the corrected linear basis visibilities is:

\[ \mathbf{M}_{ik} = \begin{bmatrix} C_{X,i} C_{Y,k} & C_{X,i} S_{Y,k} & S_{X,i} C_{Y,k} & S_{X,i} S_{Y,k} \\ C_{X,k} C_{Y,i} & C_{X,k} S_{Y,i} & S_{X,k} C_{Y,i} & S_{X,k} S_{Y,i} \\ S_{Y,i} C_{X,k} & S_{Y,i} S_{X,k} & C_{Y,i} C_{X,k} & C_{Y,i} S_{X,k} \\ S_{Y,k} C_{X,i} & S_{Y,k} S_{X,i} & C_{Y,k} C_{X,i} & C_{Y,k} S_{X,i} \end{bmatrix} \]

where * denotes the complex conjugate.

The model visibility is given by:

\[ \mathbf{V}_{\text{model}} = \mathbf{M}_{ik} \mathbf{S}_{\text{C}} \]

The components of the model visibility vector in the linear basis, \( \mathbf{V}_{\text{model}} \), are:

\[ V_{X,ik} = (S_0 M_{0,0} + S_1 M_{0,1} + S_2 M_{0,2} + S_3 M_{0,3}) g_{x,i} g_{x,k} \]
\[ V_{Y,ik} = (S_0 M_{1,0} + S_1 M_{1,1} + S_2 M_{1,2} + S_3 M_{1,3}) g_{y,i} g_{y,k} e^{j\delta} \]
\[ V_{X,Y,ik} = (S_0 M_{2,0} + S_1 M_{2,1} + S_2 M_{2,2} + S_3 M_{2,3}) g_{x,i} g_{y,k} e^{-j\delta} \]
\[ V_{Y,Y,ik} = (S_0 M_{3,0} + S_1 M_{3,1} + S_2 M_{3,2} + S_3 M_{3,3}) g_{y,i} g_{y,k} \]

Partial derivatives

The nonlinear fitting routines need the first and second partial derivatives of Eq. 7 wrt each of the parameters being fitted. The parameters which may be adjusted and for which partial derivatives may be needed are:

- \( ipol \) Source Stokes I.
- \( qpol \) Source Stokes Q.
- \( upol \) Source Stokes U.
- \( vpol \) Source Stokes V.
- \( \phi_X \) Orientation of X feed antenna \( i \).
- \( \phi_Y \) Orientation of Y feed antenna \( i \).
- \( \theta_X \) Ellipticity of X feed antenna \( i \).
- \( \theta_Y \) Ellipticity of Y feed antenna \( i \).
- \( g_x \) Gain correction to X feed antenna \( i \).
- \( g_y \) Gain correction to Y feed antenna \( i \).
- \( \delta \) Phase difference between X and Y parallel systems.

Source derivatives, all second derivatives are 0.

\[ \frac{\partial V_{XX}}{\partial ipol} = \begin{bmatrix} C_{X,i} C_{Y,k} & C_{X,i} S_{Y,k} & S_{X,i} C_{Y,k} & S_{X,i} S_{Y,k} \end{bmatrix} g_{x,i} g_{x,k} \]

\[ \frac{\partial V_{XX}}{\partial qpol} = \begin{bmatrix} C_{X,i} C_{Y,k} & C_{X,i} S_{Y,k} & S_{X,i} C_{Y,k} & S_{X,i} S_{Y,k} \end{bmatrix} g_{x,i} g_{x,k} \]

\[ \frac{\partial V_{XX}}{\partial upol} = \begin{bmatrix} C_{X,i} C_{Y,k} & C_{X,i} S_{Y,k} & S_{X,i} C_{Y,k} & S_{X,i} S_{Y,k} \end{bmatrix} g_{x,i} g_{x,k} \]

\[ \frac{\partial V_{XX}}{\partial vpol} = \begin{bmatrix} C_{X,i} C_{Y,k} & C_{X,i} S_{Y,k} & S_{X,i} C_{Y,k} & S_{X,i} S_{Y,k} \end{bmatrix} g_{x,i} g_{x,k} \]

The model of the observed visibility is then

\[ \mathbf{V}_{\text{model,ik}} = \mathbf{M}_{ik} \mathbf{S}_{\text{C}} \]

where \( \delta \) denotes the complex conjugate.

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\[ \frac{\partial V_{X,Y}}{\partial \phi_{X,i}} = \left( -j S_0 M_{0,0} - j S_1 M_{0,1} \right) \]
\[ \quad + j S_2 M_{0,2} + j S_3 M_{0,3} \]
\[ \frac{g_x, g_y}{\partial \phi_{X,i}} = -V_{X,X} \]
\[ \frac{\partial^2 V_{X,X}}{\partial \phi_{X,i}^2} = -V_{X,X} \]
\[ \frac{\partial V_{X,Y}}{\partial \phi_{X,i}} = \left( -j S_0 M_{1,0} - j S_1 M_{1,1} \right) \]
\[ \quad + j S_2 M_{1,2} + j S_3 M_{1,3} \]
\[ \frac{g_x, g_y e^{i\delta}}{\partial \phi_{X,i}} = -V_{X,Y} \]
\[ \frac{\partial^2 V_{X,Y}}{\partial \phi_{X,i}^2} = -V_{X,Y} \]

Non zero derivatives wrt \( \phi_H \):
\[ \frac{\partial V_{X,Y}}{\partial \phi_H} = \left( + j S_0 M_{1,0} - j S_1 M_{1,1} \right) \]
\[ \quad + j S_2 M_{1,2} - j S_3 M_{1,3} \]
\[ \frac{g_x, g_y e^{i\delta}}{\partial \phi_H} = -V_{X,Y} \]
\[ \frac{\partial^2 V_{X,Y}}{\partial \phi_H^2} = -V_{X,Y} \]
Non zero derivatives wrt $\theta$:

$$\frac{\partial V_Y}{\partial \theta_{Y_i}} = j(-S_0 S_Y^* C Y_i + S_1 S_Y^* C Y_i - S_2 Y_i^* S Y_i - S_3 C Y_i S Y_i^*)$$

$$g_{Y_i}, g_{Y_k} e^{i\delta}$$

$$\frac{\partial^2 V_Y}{\partial^2 \theta_{Y_i}} = -V_Y$$

$$\frac{\partial V_{XY}}{\partial \theta_{Y_i}} = j(-S_0 C X_i C Y_i + S_1 C X_i S Y_i - S_2 S X_i^* C Y_i^* - S_3 S X_i S Y_i)$$

$$g_Y, g_{Y_k} e^{i\delta}$$

$$\frac{\partial^2 V_{XY}}{\partial^2 \theta_{Y_i}} = -V_{XY}$$

Derivatives wrt $g_X$, second derivatives are 0.0:

$$\frac{\partial V_{XX}}{\partial g_{x,i}} = (S_0 M_{0,0} + S_1 M_{0,1} + S_2 M_{0,2} + S_3 M_{0,3}) g_{x,k}$$

$$\frac{\partial V_{XX}}{\partial g_{x,k}} = (S_0 M_{1,0} + S_1 M_{1,1} + S_2 M_{1,2} + S_3 M_{1,3}) g_{x,i}$$

$$\frac{\partial V_{XY}}{\partial g_{x,i}} = (S_0 M_{2,0} + S_1 M_{2,1} + S_2 M_{2,2} + S_3 M_{2,3}) g_{Y_i} e^{i\delta}$$

$$\frac{\partial V_{XY}}{\partial g_{x,k}} = (S_0 M_{3,0} + S_1 M_{3,1} + S_2 M_{3,2} + S_3 M_{3,3}) g_{Y_i} e^{-i\delta}$$

Derivatives wrt $g_Y$, second derivatives are 0.0:

$$\frac{\partial V_{YY}}{\partial g_{y,i}} = (S_0 M_{0,0} + S_1 M_{0,1} + S_2 M_{0,2} + S_3 M_{0,3}) g_{x,i} e^{i\delta}$$

$$\frac{\partial V_{YY}}{\partial g_{y,k}} = (S_0 M_{1,0} + S_1 M_{1,1} + S_2 M_{1,2} + S_3 M_{1,3}) g_{x,k} e^{-i\delta}$$

$$\frac{\partial^2 V_{YY}}{\partial^2 g_{y,i}} = -V_{YY}$$

$$\frac{\partial^2 V_{YY}}{\partial^2 g_{y,k}} = -V_{YY}$$

Derivatives of $\chi^2$ wrt parameters

The $\chi^2$ of the fit is defined as

$$\chi^2 = \sum_{i=0}^{n} (model_i - obs_i)^2 / \sigma_i^2$$

where $model_i$ is the model value for observation $i$ (real or imaginary part of visibility), $obs_i$ is the observation for $i$ and $\sigma_i^2$ is the variance of the amplitude of observation $i$. Denote $model_i - obs_i$ as $resid_i$.

The derivative of $\chi^2$ wrt each parameter $p$ is:

$$\frac{\partial \chi^2}{\partial p} = \sum_{i=0}^{n} 2 (resid_i \frac{\partial model_i}{\partial p}) / \sigma_i^2$$

This expression is complex, to get the real value, instead use

$$\frac{\partial \chi^2}{\partial p} = \sum_{i=0}^{n} 2 \left( \text{Real}(resid_i) \text{Real}(\frac{\partial model_i}{\partial p}) + \text{Imag}(resid_i) \text{Imag}(\frac{\partial model_i}{\partial p}) \right) / \sigma_i^2$$

where Real and Imag denote the real and imaginary parts of the argument.

The second derivative of $\chi^2$ wrt each parameter $p$ is:

$$\frac{\partial^2 \chi^2}{\partial p^2} = \sum_{i=0}^{n} 2 \left( \frac{\partial model_i}{\partial p} \frac{\partial model_i}{\partial p} + resid_i \frac{\partial^2 model_i}{\partial p^2} \right) / \sigma_i^2$$

This expression is also complex, to get the real value, instead use

$$\frac{\partial^2 \chi^2}{\partial p^2} = \sum_{i=0}^{n} 2 \left( \text{Real}(\frac{\partial model_i}{\partial p}) \text{Real}(\frac{\partial model_i}{\partial p}) + \text{Imag}(\frac{\partial model_i}{\partial p}) \text{Imag}(\frac{\partial model_i}{\partial p}) \right) + \text{Real}(resid_i) \text{Real}(\frac{\partial^2 model_i}{\partial p^2}) + \text{Imag}(resid_i) \text{Imag}(\frac{\partial^2 model_i}{\partial p^2}) / \sigma_i^2$$
REFERENCES