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Three-Phase Switching with m-Sequences for Sideband Separation in Radio Interferometry

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ABSTRACT. Orthogonal sequences known as m-sequences can be used in place of Walsh functions in phase switching and sideband separation in cross-correlation interferometers. Functions based on three-character m-sequences may be advantageous because they may provide a larger set of mutually orthogonal modulation and demodulation functions, and hence support a larger number of antennas for a given sequence length, than allowed by other orthogonal sequences such as Walsh functions. The reason for this advantage is that if the demodulation functions are formed from the differences of the three-character m-sequence modulation functions, then because the m-sequences obey an addition rule whereby sums or differences of m-sequences are also m-sequences, the demodulation functions are also members of the original orthogonal set. In a complete set of sequences, all the differences are of course duplicates of the original sequences. However, certain subsets of sequences have differences which are not members of the subset, and these subsets can be used to form modulation and demodulation functions which have both the desired uniqueness and orthogonality properties. While it is not obvious how to select the subsets, heuristic methods seem reasonably successful.

1. INTRODUCTION

Phase switching in radio interferometry refers to the technique of periodically reversing the phase of the signal from one antenna of a pair and simultaneously reversing the sign of the correlation of the signals from the pair. Originally introduced by Ryle (1952) as a means of multiplying the voltages from the two antennas, the technique is still used in modern interferometers to reduce noise and DC drifts. Sideband separation refers to the technique of periodically introducing a 90° phase shift into the first local oscillator (LO) at the receivers to form both in-phase and quadrature signals. These signals may be later combined to separate the signals from the upper and lower sidebands of the first downconversion.

While it is easy to see how to phase switch a two-element interferometer, hold one phase constant and switch the other, the switching functions become more interesting when more than two antennas are involved. For example, it is not sufficient to switch a third antenna with either function used for the first two (constant or periodic) if we want to phase switch all three possible correlations. The requirement of sideband separation adds further complexity. Modern interferometers have employed Walsh functions as modulation functions for both phase switching and sideband separation (Wright et al. 1973; Granlund, Thompson, & Clark 1978). The Walsh functions are orthogonal binary functions, analogous to the better known sine and cosine orthogonal functions (Harmuth 1970).

An additional complication arises in making a set of demodulation functions which are unique, orthogonal to each other, and orthogonal to each of the modulation functions as well. This orthogonality reduces the possibility of spurious pickup or cross talk of the switching signals. In general, the demodulation functions are linear combinations of the modulation functions and are neither orthogonal nor unique. However, in a sufficiently large set of modulation functions, one can usually select a smaller subset whose demodulation functions have these properties (Urry, Thornton, & Hudson 1985; Welch et al. 1996; Urry 1999). But if many antennas are to be supported, the length of the functions and hence the length of the switching cycle can become long. One would like a short switching cycle because this is the shortest timescale on which the phase of each antenna may be corrected. For example, it is possible to imagine that a water vapor radiometer operating in real time at each antenna could be used to correct phase errors due to the atmosphere on a timescale of a fraction of a second. Thus it is an interesting question as to how to provide the largest set of mutually orthogonal modulation and demodulation functions with the shortest length.

Orthogonal functions known as m-sequences suggest themselves for this problem because they obey an addition rule whereby the sum or difference of a pair of m-sequences is also an m-sequence and hence a member of the original orthogonal set. Because it is possible to make the demodulation functions from simple differences of the modulation functions, the m-sequence demodulation functions are then also members of the same orthogonal set. The question then becomes one of finding the largest subset for a given sequence length whose differences are both unique and not members of the subset.

In order to take advantage of the addition property of the m-sequences, it is necessary to use a method which switches between three phases, rather than between four phases as is done with Walsh functions. This is because the m-sequences follow the addition rules for Galois fields, and these rules are the same as for the addition of phases only for Galois fields with a prime number of elements. The four-phase switching is naturally suggested by the method of sideband separation which employs the in-phase and quadrature (separated from the in-phase by 90°) components of the signal. In this four phase method, one first subtracts the correlations which are collected with phases 180° apart. This cancels any drifts in the signal and results in two data streams with phases separated by 90°. These two data streams can then be combined to produce the data from the upper and lower sidebands separately. However, one may also perform both the operations of drift cancellation and sideband separation using a method which switches between three phases separated by 120°. In this case the in-phase and quadrature signals are formed from linear combinations of the three-phase data analogous to forming two vectors separated by 90° by addition of three vectors separated by 120°. The addition should be done so that equal weight is applied to each data stream and the noise of the sums are then averaged down as much as possible.

2. PHASE SWITCHING USING WALSH FUNCTIONS

In order to illustrate the method of phase switching and sideband separation we describe a simple technique, employed for example at the Owens Valley Radio Observatory. This technique uses two sets of Walsh functions, the first for the 90° sideband separation and the second for the 180° phase switching. The phase switching cycles are nested within the sideband separation cycles so that within each of the time steps during which the first LO is in either the 0° or 90° phase of the sideband separation cycle, the first LO is switched by 180° through a complete cycle of another Walsh function. For example, if we use Walsh functions with a length of eight steps, then a complete cycle of both the 90° and 180° switching will involve 64 steps.

With this switching arrangement, the data may be processed as follows. The correlation of the signals from two antennas will contain a factor of the cosine of the difference in phases between the LOs of two antennas. For example,

$$\cos\,(\Delta\psi_{ij} + \omega_{\rm IF}\,\tau + \theta_i - \theta_j)\ , \tag{1}$$

where $\Delta\psi_{ij}$ is the phase difference of the signals into antennas i and j (this is the signal to be measured), $\omega_{\rm IF}$ is the IF or intermediate frequency after downconversion, τ is the time lag of the correlation, and θ_i and θ_j are the phases of the LOs of the two antennas.

We first resolve the 180° phase switching cycle which is the inner nest of the two cycles. The phase difference due to this cycle will be either 0° or 180°, and in computing the correlation, we sum the signals with a positive or negative sign according to the sign of the cosine factor. This sequence of positive and negative ones formed from the difference of the second set of Walsh functions is the first part of the demodulation function. With the 180° phase switching resolved we are left with two streams of data representing the two states arising from the 90° switching of the first set of Walsh functions. The difference of these first Walsh functions is the second part of the demodulation function. Because $\cos\,(a - \pi/2) = \sin\,(a)$ we have two data streams which represent the in-phase and quadrature components of the measured signal.

Taking the Fourier transform of both these data streams results in

$$\mathrm{FT}[C_{ij}(\tau)] = U_{ij}(\nu)\,\exp\,(-i\Delta\psi^u_{ij}) + L_{ij}(\nu)\,\exp\,(i\Delta\psi^l_{ij})\ , \tag{2}$$

$$\mathrm{FT}[S_{ij}(\tau)] = U_{ij}(\nu) i\,\exp\,(-i\Delta\psi^u_{ij}) - L_{ij}(\nu) i\ \exp\,(i\Delta\psi^l_{ij})\ , \tag{3}$$

where $C_{ij}(\tau)$ is the in-phase correlation, $S_{ij}(\tau)$ is the quadrature correlation, $U_{ij}(\nu)$ and $L_{ij}(\nu)$ are the products of the amplitudes of the upper and lower sideband signals from antennas i and j, and the terms $\Delta\psi_{ij}$ are the phase differences of the incoming signals with the superscripts u and l indicating the sideband. The equation for $\mathrm{FT}[S_{ij}(\tau)]$ can be recognized through Euler's identity as a $\pi/2$ shift of the equation for $\mathrm{FT}[C_{ij}(\tau)]$. In these equations, the negative frequencies contain no independent information since the measured correlations are purely real functions, and the Fourier transform of a purely real function is symmetric. Since we use only the positive frequencies, multiplying $\mathrm{FT}[S_{ij}(\tau)]$ by i is equivalent to taking the Hilbert transform of $S_{ij}(\tau)$ followed by the Fourier transform. The sidebands may now be separated as

$$U_{ij}(\nu)\,\exp\,[-i\Delta\psi^u_{ij}(\nu)] = \tfrac{1}{2}\{\mathrm{FT}[C_{ij}(\tau)] - i\mathrm{FT}[S_{ij}(\tau)]\}\ , \tag{4}$$

$$L_{ij}(\nu)\,\exp\,[i\Delta\psi^\ell_{ij}(\nu)] = \tfrac{1}{2}\{\mathrm{FT}[C_{ij}(\tau)] + i\mathrm{FT}[S_{ij}(\tau)]\}\ . \tag{5}$$

The left-hand sides of equations (4) and (5) are the desired upper and lower sideband visibility amplitudes and phases for the correlation of antennas i and j.

Walsh functions can be used in a different method of sideband separation and phase switching which does not require nesting. In this method, employed many years ago at the Green Bank interferometer (B. Clark 1999, private communication) and currently at the Hat Creek Observatory (Urry et al. 1985; Welch et al. 1996; Urry 1999), the in-phase and quadrature signals are generated by switching the first LO by 90° while the 180° signals for phase switching are generated by the second LO. In this method, the first and second LOs are switched simultaneously and two independent Walsh functions are required for each receiver to be correlated. In this method, the demodulation functions are also linear combinations of Walsh functions, and certain subsets of Walsh functions are used to form unique and orthogonal demodulation functions.

3. SWITCHING WITH SHIFTED m-SEQUENCES

The m-sequences, short for "maximal-length shift register sequences," are sequences of some number of characters which have properties analogous to those of the better known two-character pseudorandom binary sequences (Zierler 1959). In particular, the complex autocorrelation function of both pseudorandom binary sequences and the multicharacter m-sequences is equal to unity at zero lag and equal to the inverse of the length of the sequence at all other lags. This property implies that a set of m-sequences which are derived by cyclically shifting a single m-sequence are nearly orthogonal to each other. It is a simple matter to make a set of functions based on m-sequences which are orthogonal and suitable for sideband separation and phase switching in radio interferometry.

An m-sequence is obtained from a recurrence relation,

$$a_{i+m} = -h_{m-1}\, a_{i+m-1} - h_{m-2}\, a_{i+m-2} - \cdots - h_1\, a_{i+1} - h_0\, a_i \;, \tag{6}$$

where

$$h(x) = x^m + h_{m-1}\, x^{m-1} + \cdots + h_1\, x + h_0 \tag{7}$$

is a primitive polynomial, $h \neq 0$, and h and a are elements of a Galois field of q elements denoted GF(q). The recurrence relation will generate an infinite sequence of period $q^m - 1$ of which any segment of length $q^m - 1$ is an m-sequence. For example, take the case of the Galois field of four elements, GF(4) with the primitive polynomial $h(x) = x^2 + x + \omega$ of degree $m = 2$. The four elements 0, 1, ω, ω^2 satisfy $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$ so ω is a cube root of unity (Balza, Fromageot, & Maniere 1967). We obtain the m-sequence

$$0\ 1\ 1\ \omega^2\ 1\ 0\ \omega\ \omega\ 1\ \omega\ 0\ \omega^2\ \omega^2\ \omega\ \omega^2\ . \tag{8}$$

As another example, for the Galois field of three elements, we have the following sequences of lengths $3^m - 1$, with $m = 2$ and 3 (Godfrey 1966),

$$0\ 1\ 2\ 2\ 0\ 2\ 1\ 1\ , \tag{9}$$

$$00101211201110020212210222\ . \tag{10}$$

As with the binary pseudorandom sequences, the autocorrelation function of the m-sequences is the autocorrelation of the complex sequence obtained by replacing each of the elements in an m-sequence by $s_j = \exp\,[2\pi i r/q]$ where r is the number of the element in GF(q).

$$c(j) = \frac{1}{n} \sum_{i=0}^{n-1} s_i \overline{s_{i+j}}\ ,$$
$$c(0) = 1\ ,$$
$$c(j) = -\frac{1}{q^m - 1}\ . \tag{11}$$

The following combination of properties relevant to our application is equivalent to the autocorrelation property:

i) In an m-sequence of length $q^m - 1$, each nonzero element occurs q^{m-1} times and the element 0 occurs $q^{m-1} - 1$ times.

ii) In the terms of the autocorrelation function, the lags or shifts which are not multiples of $(q^{m-1} - 1)/(q - 1)$ contain each pair of elements q^{m-2} times except the pair $\{0, 0\}$, which occurs $q^{m-2} - 1$ times.

iii) For shifts of $k(q^{m-1} - 1)/(q - 1)$ there are q^{m-1} pairs of the nonzero elements $\{\alpha, \beta^j\alpha\}$, where β is a primitive of the Galois field, while the pair $\{0, 0\}$ occurs $q^{m-1} - 1$. In the case of GF(4) the last part of this statement means that for the sequence of length 15, the pairs $\{1, \omega\}$, $\{1, \omega^2\}$, $\{\omega, \omega^2\}$ each occur four times while the pair $\{0, 0\}$ occurs three times.

These properties suggest that the set of $q^m - 1$ cyclic shifts of an m-sequence together with the sequence of zeroes, **0**, may be used as the switching functions in interferometry if we add to each sequence an additional zero in the same place, for example at the beginning, of each of the already shifted sequences. This additional zero makes each sequence q^m states long, and with the sequence of all zeroes, there are q^m different sequences.

There are a number of ways to use the m-sequences. The binary m-sequences, that is, the pseudo-random sequences consisting of zeroes and ones, may simply be substituted for the Walsh functions. Alternatively, the four-character sequence over GF(4) may be used to set the four phases at the first LO, and the data processed as with the method of nested Walsh functions. With the four-character m-sequence, because of property (iii), we can use only shifts up to $k(q^{m-1}-1)/(q-1)$, or alternatively one can use all q^m of the sequences if we double the sequence length by appending to each sequence a copy of the sequence, but with the 1's and ω's swapped. Applying the m-sequences in these two methods does not seem to improve on the methods using Walsh functions. The reason in the case of the four-character m-sequence is that the demodulation functions which are differences of the m-sequence modulation functions are computed as differences modulo 4 and not according the addition rules of the Galois field GF(4). Thus the resulting demodulation functions are not m-sequences and not members of the original orthogonal set.

However, in the case of the three-character m-sequences the addition rules for GF(3) are simply modulo 3. Therefore the demodulation functions formed as differences are also m-sequences and orthogonal to the modulation functions as well as the other demodulation functions. Finding a suitable set of modulation functions amounts to finding a subset of sequences whose differences contain no duplicates.

Tables 1 and 2 list examples of suitable subsets found by heuristic search methods. In the tables, the sequence labeled "0" means the original sequence, and the sequence labeled "1" means the original sequence shifted to the right by one character. The sequence labeled q^m is the sequence of zeroes. The number at the intersection is the sequence number of the difference of the sequences indicated at the start of the row and column. Because the difference of sequences $a-b$ is not the same sequence as the difference of $b-a$, the differences must be taken as given by the (row − column). The case of the sequences of length 243 is particularly interesting because the selected subset and its differences make up 231 out of the total of 243 orthogonal functions. Rephrased for interferometry, this subset can provide unique orthogonal functions for the 210 baselines demodulation functions as well as the 21 antenna modulation functions.

If it is not required to make the demodulation functions unique from the modulation functions, but only unique and orthogonal to the other demodulation functions, one can in general do a little better. For example, Table 3 lists such a subset of 11 sequences of length 81.

Although phase switching between four phases is naturally suggested in order to cancel drifts and separate the sidebands, it is also possible to perform these functions with data obtained by switching between three phases separated by 120° as would be required by the three-character

TABLE 1

A SUBSET OF 10 SEQUENCES OF LENGTH 81

	80	42	77	72	64	54	49	69	60	11
11	51	39	5	46	12	50	73	27	48	
60	20	75	4	23	21	65	55	56		
69	29	70	43	41	10	52	59			
49	9	25	36	74	7	35				
54	14	45	79	47	34					
64	24	0	68	38						
72	32	22	58							
77	37	31								
42	2									
80										

TABLE 2

A SUBSET OF 21 SEQUENCES OF LENGTH 243

	242	24	202	199	146	107	166	44	208	116	58	115	112	217	175	75	149	168	190	89	74
74	195	180	65	142	193	238	150	133	181	34	103	49	230	104	201	79	223	95	191	39	
89	210	186	59	72	21	29	117	60	12	224	155	109	170	80	225	102	200	70	216		
190	69	33	66	235	23	240	4	110	130	41	173	143	51	83	19	207	3	136			
168	47	187	154	144	114	125	119	9	231	52	162	172	22	140	204	124	86				
149	28	213	85	184	40	188	8	158	20	159	171	222	17	148	169	0					
75	196	206	92	161	63	129	67	141	37	50	38	138	101	48	27						
175	54	167	68	134	139	227	90	11	97	108	1	105	73	135							
217	96	189	46	18	13	211	106	218	132	122	229	194	226								
112	233	94	16	212	98	55	174	164	32	232	241	127									
115	236	137	215	219	91	53	176	153	43	120	111										
58	179	131	100	205	7	30	152	71	163	156											
116	237	221	10	128	84	31	151	42	192												
208	87	64	61	123	57	113	5	26													
44	165	182	185	178	2	126	234														
166	45	118	36	88	62	99															
107	228	157	239	183	209																
146	25	93	177	82																	
199	78	56	214																		
202	81	76																			
24	145																				
242																					

TABLE 3

A Subset of 11 Sequences of Length 81

	72	22	8	4	64	23	66	24	60	49	50
50	48	49	32	76	18	51	3	29	30	53	
49	74	50	61	78	7	28	73	71	15		
60	23	44	47	62	21	19	31	33			
24	43	79	1	14	24	27	8				
66	37	35	64	59	41	22					
23	60	26	6	58	36						
64	38	46	66	34							
4	75	77	45								
8	65	56									
22	52										
72											

m-sequences. In this three-phase method, the phase differences of the LO signals at the two antennas are either 0° or $\pm 120°$. One can visualize the three data streams with these phase differences as analogous for example to the three unit vectors from the origin $\boldsymbol{e}_1 = (0,\ 1)$, $\boldsymbol{e}_2 = (\sqrt{3}/2,\ -1/2)$, $\boldsymbol{e}_3 = (-\sqrt{3}/2,\ -1/2)$. This analogy can be seen by writing the equation for the correlation of two signals following multiplication by the LO signals in the following form,

$$
\begin{aligned}
C_{ij}(\tau) = {} & \cos(-\phi_j + \phi_i)\int_0^\infty U_{ij}\cos(\omega_{\rm IF}\tau + \Delta\psi^u_{ij})dv \\
& - \sin(-\phi_j + \phi_i)\int_0^\infty U_{ij}\sin(\omega_{\rm IF}\tau + \Delta\psi^u_{ij})dv \\
& + \cos(\phi_j - \phi_i)\int_0^\infty L_{ij}\cos(\omega_{\rm IF}\tau - \Delta\psi^\ell_{ij})dv \\
& - \sin(\phi_j - \phi_i)\int_0^\infty L_{ij}\sin(\omega_{\rm IF}\tau - \Delta\psi^\ell_{ij})dv\,. \qquad (12)
\end{aligned}
$$

Here the differences of the LO phases appear as coefficients on the sine and cosine terms of the upper and lower sideband data, and the three data streams with different phase differences can be added or subtracted in the same way as vectors with those coefficients. In particular since the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ make up a frame or set of non-orthogonal basis vectors for the two-dimensional plane, these vectors can be combined to form two vectors 90° apart. Similarly, the data streams analogous to these vectors can also be combined to form the required in-phase and quadrature components. But first, in order to cancel drifts or constant offsets, we can subtract each of the three pairs of correlation data. In terms of the three basis vectors, we form the differences $\boldsymbol{e}_1 - \boldsymbol{e}_2$, $\boldsymbol{e}_2 - \boldsymbol{e}_3$, and $\boldsymbol{e}_3 - \boldsymbol{e}_1$. These differences are three other vectors of equal length also separated by 120° which may also serve as basis vectors, but in the data, constant offsets have been subtracted out. There are many ways to combine these difference vectors to form the 90° vectors, but in order to maximize the signal-to-noise ratio of the data, the linear combinations of the data streams for each of the two sidebands should be formed so that at the end of all the arithmetic, the three original data streams are combined with equal weights applied to each stream. This follows from the general principle that to best average a set of *n* measurements, one should, in the absence of any other information, add the measurements together and divide by *n*. For example using the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ one can form two vectors separated by 90°,

$$
\begin{aligned}
\boldsymbol{v}_1 &= \sqrt{2}/2\,,\ \sqrt{2}/2 \\
&= (\sqrt{2}/4)\boldsymbol{e}_1 \\
&\quad + 1/2(\sqrt{2}/\sqrt{3} - \sqrt{2}/2)\boldsymbol{e}_2 \\
&\quad - 1/2(\sqrt{2}/\sqrt{3} + \sqrt{2}/2)\boldsymbol{e}_3\,, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{v}_2 &= -\sqrt{2}/2\,,\ \sqrt{2}/2 \\
&= (\sqrt{2}/4)\boldsymbol{e}_1 \\
&\quad - 1/2(\sqrt{2}/\sqrt{3} + \sqrt{2}/2)\boldsymbol{e}_2 \\
&\quad + 1/2(\sqrt{2}/\sqrt{3} - \sqrt{2}/2)\boldsymbol{e}_3\,, \qquad (14)
\end{aligned}
$$

whose sum contains the three vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ in equal weight. Analogously by equations (4) and (5), the upper and lower sideband data can be found from the sum and differences of the Fourier transforms of linear combinations of the differenced data streams.

REFERENCES

Balza, C., Fromageot, A., & Maniere, M. 1967, Electron. Lett., 3, 313

Godfrey, K. R. 1966, Electron. Lett., 2, 241

Granlund, J., Thompson, A. R., & Clark, B. G. 1978, IEEE Trans. Electromagn. Comput., EMC-20, 451

Harmuth, H. F. 1970, Transmission of Information by Orthogonal Functions (Berlin: Springer)

Ryle, M. A. 1952, Proc. R. Soc. A, 211, 351

Urry, W. L. 1999, On the Use of Shifted m-Sequences for Phase Switching or Why Not Complex Walsh Functions (BIMA Tech. Memo 76; Univ. California, Berkeley, Dept. Astron.)

Urry, W. L., Thornton, D. D., & Hudson, J. A. 1985, PASP, 97, 745

Welch, W. J., et al. 1996, PASP, 108, 93

Wright, M. C. H., Clark, B. G., Moore, C. H., & Coe, J. 1973, Radio Sci., 8, 763

Zierler, N. 1959, J. Soc. Indus. Appl. Math., 7, 31