ERRORS IN RADIO HOLOGRAPHY ON A SPHERICAL SURFACE IN THE FRESNEL ZONE

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ABSTRACT.

INTRODUCTION

Radio "holography" is frequently used in the measurement and adjustment of large radiotelescopes. A strong radiation source is needed - very often a satellite beacon is used - and the surface profile of the radiotelescope is derived by a simple Fast Fourier transform of the far field radiation pattern.

However in many cases it is convenient to use a ground based transmitter and the radiation pattern is naturally measured over a spherical surface. In the case of large radiotelescopes this is inevitably in the near field region and allowance for this is normally made by a Fresnel correction to the aperture plane phase distribution. This has the big advantage of retaining the Fast Fourier transform algorithm. It is particularly important if "phase retrieval" holography is used since the phase retrieval algorithms are computationally intensive. The question thus arises of what are the limits to the accuracy which can be achieved by retaining the Fresnel correction ?

McGrane (1983) has already considered this question for the case where the far field radiation pattern is to be determined from near field measurements over a spherical surface. From one dimensional simulations he concluded that the method was viable for distances greater than 0.2 times the Rayleigh distance, defined as $D^2/2\lambda$. Where D is the telescope diameter and λ is the wavelength.

The limits to the Fresnel approximation for beam propagation between two planes has been studied in the optical range by Southwell (1981). He found that within the beam itself the Fresnel approximation leads to phase errors less than 0.02 radians but that outside this central region the phase errors increased.

None of these works gives guidance for the specific problem of radioholography. The present paper attempts to assess the limits in this case and to clarify the choice of the parameters of the experimental setup - source distance, wavelength, size of data window etc.

THE SCALAR DIFFRACTION INTEGRAL.

The radiation from an aperture can be calculated by the scalar diffraction integral under certain conditions.These are that the field in the aperture has the electric and magnetic components related locally as in a plane wave (geometrical optics approximation) and that edge effects at the boundary can be neglected. Then, in the radiation zone, eq.122 of Silver (1984) can be written as

$$E_{nf} = \frac{jk}{4\pi} \int \int \frac{E}{r} e^{-jkr} [\mathbf{n}.\mathbf{r_1} + \mathbf{n}.\mathbf{s}] dS \tag{1}$$

where the scalar amplitude E_p at the field point is expressed as an surface integral of the amplitude E over the aperture. Here $\mathbf{n}$ is a unit vector normal to the surface, $\mathbf{s}$ is a unit vector defining the ray vector (normal to the phase front) in the aperture, and $\mathbf{r_1}$ is the unit vector along r joining the element of area dS to the field point. As usual k is the propagation constant $\frac{2\pi}{\lambda}$.

THE PARAXIAL APPROXIMATION.

In the present case of a telescope focussed at infinity and where the integration is made over the planar aperture $\mathbf{n}.\mathbf{s} = 1$. For the "paraxial approximation" it is assumed that the "inclination factor" $[\mathbf{n}.\mathbf{r_1} + 1]$ can be considered constant over the aperture, and that the radius r in the denominator is similarly constant. The exponential term is also approximated as follows for measurements made over a spherical surface.

If the field point lies on a sphere of radius R, centered on the planar aperture, then it can be specified by direction cosines u,v and the aperture point can similarly be given coordinates x,y. Then r becomes

$$r = \left[(Ru-x)^2 + (Rv-y)^2 + R^2(1-u^2-v^2)\right]^{0.5}$$

$$\frac{r}{R} = \left[1 + \frac{x^2}{R^2} + \frac{y^2}{R^2} - 2\frac{ux}{R} - 2\frac{vy}{R}\right]^{0.5} \tag{2}$$

on expanding using the binomial theorum

$$\frac{r}{R} = 1 - \frac{(ux+vy)}{R} + \frac{x^2+y^2}{2R^2} - \frac{1}{8}\left(\frac{x^2+y^2}{R^2}\right)^2 + \frac{\epsilon}{R} \tag{3}$$

where the error term in the optical path ϵ is

$$\epsilon = \frac{1}{2R^2}(x^2+y^2)(ux+vy) - \frac{1}{2R}(ux+vy)^2 + ... \tag{4}$$

In the far field only the first two terms of equation (3) need be retained, when eq(1) becomes a Fourier Transform. In the Fresnel zone term 3 of eq(3) is normally retained and, as it is not a function of u,v, does not disturb the Fourier Transform relationship, it merely provides a defocussing. In a similar way it is possible to retain the 4'th term - as a correction, in the form of spherical aberration, to the Fresnel defocussing term. However, the remaining terms of eq(3) cannot be incorporated in the Fourier Transform and constitute errors in the Fresnel calculation of the diffracted field.

The expression for the paraxial approximation to the scalar diffraction integral becomes

$$E_{nf} = A \int\int E e^{-jk\left[\frac{x^2+y^2}{2R} - \frac{1}{8R^3}\left(x^2+y^2\right)^2\right]} e^{jk(ux+vy)} e^{-jk\epsilon} dS \tag{5}$$

where

$$A = \frac{jk}{2\pi R} e^{-jkR} \tag{6}$$

which can be recognised as the Fourier Transform of the defocussed aperture distribution $E e^{-jk\left[\frac{x^2+y^2}{2R} - \frac{1}{8R^3}\left(x^2+y^2\right)^2\right]}$ with a correction factor $e^{-jk\epsilon}$ (normally ignored).

An upper limit to the error ϵ_m in the optical path can be obtained by substituting extreme values x_m,y_m and u_m,v_m, subject to the constraints imposed by the circular outline of the aperture. The result is particularly simple if the near field data window is also circular.

$$\epsilon_m = \frac{x_m^3 u_m}{2R^2} + \frac{x_m^2 u_m^2}{2R} \tag{7}$$

For a square data window this becomes

$$\epsilon_m = \frac{x_m^3 u_m}{\sqrt{2}R^2} + \frac{x_m^2 u_m^2}{R} \tag{8}$$

In the near field, if measurements are made with the telescope focussed normally, measurements must be made to at least values of $u_m, v_m \geq \pm\frac{KD}{2R}$ to avoid truncation errors. In this way a data window exceeding the diameter of the geometrical optics beam by a factor K is scanned. In the aperture plane $x_m, y_m = \frac{D}{2}$, where D is the aperture diameter. Equation 6 then yields a very simple upper limit to the path length error

$$\epsilon_m = \frac{K(\sqrt{2}+2K)}{32}\frac{D^4}{R^3} \tag{9}$$

In practice the telescope may be refocussed at the distance of the transmitter, to reduce truncation, or a larger map of diameter N pixels may be scanned to obtain higher surface resolution for example. If the map is sampled at intervals of $\frac{s\lambda}{D}$ then the maximum path error becomes

$$\epsilon_m = \frac{DNs\lambda}{32R}\left[\frac{\sqrt{2}D}{R} + \frac{2Ns\lambda}{D}\right] \tag{10}$$

since by definition

$$u_m = v_m = \frac{sN\lambda}{2D} \tag{11}$$

For the special case of a telescope "perfectly" refocussed at the distance of the transmitter, the first term of equation 10 makes no contribution.

ERRORS IN THE CALCULATED APERTURE PLANE DISTRIBUTION

The actual phase errors in the derived aperture plane field distribution and thus path errors in the derived telescope figure may be much smaller than indicated by ϵ_m. An approximate analytic expression for the error in the radiation pattern can be obtained from equations 4 and 5.

$$\delta E_{nf}(u,v) = -jkA\int\int Ee^{-jk\left[\frac{x^2+y^2}{2R} - \frac{1}{8R^3}\left(x^2+y^2\right)^2\right]}e^{jk(ux+vy)}\left[\frac{1}{2R^2}(x^2+y^2)(ux+vy) - \frac{1}{2R}(ux+vy)^2\right]dS \tag{12}$$

The corresponding error $\delta E_s(r)$ in the solution for the aperture plane field follows by inverse Fourier Transform. The phase error $\delta\phi$ in the solution for the aperture field is then

$$\delta\phi(r) = \frac{Im\left[\delta E_s(r)\right]}{E_s(r)} \simeq \frac{Im\left[\delta E_s(r)\right]}{E(r)} \tag{13}$$

since both the original field E and the solution E_s are here assumed to be real.

The expression (12) for the error in the radiation pattern becomes simple if the data window over which the pattern is measured is circular. Then the problem becomes one dimensional and Fourier Transforms become Hankel Transforms .

$$\delta E_{nf}(u) = A\int_0^{D/2} E(r)e^{-jk\left[\frac{r^2}{2R} - \frac{r^4}{8R^3}\right]}\left[\frac{jkr^2u^2}{4R}[J_0(kru) - J_2(kru) + \frac{kr^3u}{2R^2}J_1(kru)]\right]2\pi rdr \tag{14}$$

The imaginary part of the error in the near field becomes

$$\begin{aligned} Im\left[\delta E_{nf}(u)\right] = A\int_0^{D/2} E(r)\Bigg[&\frac{kr^2u^2}{4R}[J_0(kru) - J_2(kru)]Cos\left[k\left(\frac{r^2}{2R} - \frac{r^4}{8R^3}\right)\right] \\ &-\frac{kr^3u}{2R^2}J_1(kru)Sin\left[k\left(\frac{r^2}{2R} - \frac{r^4}{8R^3}\right)\right]\Bigg]2\pi rdr \end{aligned} \tag{15}$$

and from eq(12), the error in the aperture phase distribution is given by the truncated (IE with finite integration limits) Hankel transform of eq(15), since this transform does not "mix" real and imaginary parts,

$$\begin{aligned} \delta\phi(r') = \frac{2\pi k^2}{E(r')}\int_0^{W/2}\int_0^{D/2} E(r)\Bigg[&\frac{kr^2u^2}{4R}[J_0(kru) - J_2(kru)]Cos\left[k\left(\frac{r^2}{2R} - \frac{r^4}{8R^3}\right)\right] \\ &-\frac{kr^3u}{2R^2}J_1(kru)Sin\left[k\left(\frac{r^2}{2R} - \frac{r^4}{8R^3}\right)\right]\Bigg]rdrJ_0(kr'u)udu \end{aligned} \tag{16}$$

Where W is the diameter of the data window.

An analogous expression may be derived for the expansion of this phase error distribution in Zernike polynomials.

The integrals over radius r in equation (16) can be performed analytically in simple cases. For example when the telescope has been "perfectly" refocussed on the transmitter so that the argument of the cosine and sin terms become zero, and for the case where a very simple aperture illumination taper is used.

IDEALLY REFOCUSSED TELESCOPE WITH PARABOLIC ILLUMINATION TAPER

In this case

$$E(r) = 1 - \frac{4ar^2}{D^2} \tag{17}$$

where a value of $a = 0.8$ corresponds to a typical taper of -14 dB.

Then eq(16) becomes

$$\delta\phi(r') = \frac{2\pi k^2 D^2}{(D^2 - 4ar'^2)} \int_0^{W/2} \int_0^{D/2} \frac{(D^2 - 4ar^2)}{D^2} \frac{kr^2u^2}{4R} [J_0(kru) - J_2(kru)] r dr J_0(kr'u) u du \tag{18}$$

by using the substitution $x = kru$ this can be simplified to

$$\begin{aligned} \delta\phi(r) = \frac{\pi k D^4}{8R(D^2 - 4ar^2)} \int_0^{W/2} \frac{1}{(kuD/2)^2} \int_0^{kuD/2} \Big[& x^3 [J_0(x) - J_2(x)] \\ & - \frac{4ax^5}{(kuD/2)^2} [J_0(x) - J_2(x)] \Big] dx J_0(kru) u du \end{aligned}$$

(19)

integrating over x analytically, and using the variable $y = kDu/2$

$$\begin{aligned} \delta\phi(r) = \frac{\pi D^2}{kR(D^2 - ar^2)} \int_0^{\pi N/2} \Big[& y J_2(y) - y^2 J_3(y) - \frac{a}{2} [(8 - y^2) J_3(y) \\ & - 6y J_4(y) + y^2 J_5(y)] \Big] J_0\left(\frac{2yr}{D}\right) dy \end{aligned}$$

(20)

the remaining integral, corresponding to the truncated Hankel transform has no simple closed form expression.

Equation 20 has been numerically integrated for several cases of interest. Figure 1 illustrates a typical result for the aperture path error distribution as a function of radius. This shows that the errors manifest as a "ringing" concentrated at the edge of the aperture. Although the peak error at the edge can be large, the root mean square (RMS) error can be much less. This has been studied as a function of data window diameter N (measured in beam widths of $\frac{\lambda}{D}$ radians) and displayed in Figure 2. These show the peak error, RMS, and amplitude weighted RMS path error as a function of N. In all cases the relation is a power law. Square law for the peak error, as expected from the second term in equation 10 (the first term makes no contibution to the imaginary part of the integrals of eq. 15). A smaller index is observed for the root mean square values. These results for the optical path errors can be summarized by :-

for the peak optical path error

$$\epsilon_m = 5166 \frac{\pi}{k^2 R} \left(\frac{N}{128} \right)^2 \tag{21}$$

for the root mean square path error

$$\epsilon_{rms} = 812 \frac{\pi}{k^2 R} \left(\frac{N}{128} \right)^{1.6} \tag{22}$$

and for the amplitude weighted root mean square path error

$$\epsilon_{wrms} = 567 \frac{\pi}{k^2 R} \left(\frac{N}{128} \right)^{1.6} \tag{23}$$

the corresponding surface errors of the telescope mirror are approximately half these values. The form of equation 21 agrees very well with the order of magnitude estimates using the second term of equation 10. However the numerical constant is about 2.5 times smaller than that predicted from eq. 10. This presumably reflects the statistical averaging over the geometrical path error which is involved in the holographic process.

REALISTIC REFOCUS

In practice any refocussing is done by axial motion of the receiver. However this introduces significant spherical aberration in addition to the desired defocus. This leaves terms in r^4 as arguments of the trigonometric functions in equations 15 and 16 and analytic integration over r in terms of commomly used functions becomes impossible. Numerical integration is then necessary.

The phase distribution in the aperture plane, for a refocus δF is, for a prime focus telescope of focal length F

$$j2k \left(\frac{r}{2F} \right)^2 \left[1 + \left(\frac{r}{2F} \right)^2 \right]^{-1} \delta F \tag{24}$$

if a refocus $\delta F = F^2/R$ is chosen, then the defocus term in r^2 disappears and the residual aperture phase distribution becomes

$$\frac{jk}{R} \left[\frac{r^4}{8} \left(\frac{1}{R^2} - \frac{1}{F^2} \right) + \frac{r^6}{32F^4} + ... \right] \tag{25}$$

The dominant term of which is that containing $1/F^2$. When the first two term (in r^4) are substituted into eq(12) the expression for the aperture plane phase error, corresponding to that of eq(16), is

$$\delta\phi(r') = \frac{2\pi k^2 D^2}{(D^2 - 4ar'^2)} \int_0^{W/2} \int_0^{D/2} \frac{(D^2 - 4ar^2)}{D^2} \left[\frac{kr^2u^2}{4R} [J_0(kru) - J_2(kru)] Cos[\frac{kr^4}{8R}(R^{-2} - F^{-2})] \right. \\ \left. - \frac{kr^3u}{2R^2} J_1(kru) Sin[\frac{kr^4}{8R}(R^{-2} - F^{-2})] \right] rdr J_0(kr'u) udu \tag{26}$$

on changing variables this becomes

$$\delta\phi(r) = \frac{\pi D^2}{2kR(D^2 - 4ar^2)} \int_0^{\pi N/2} \frac{1}{y} \int_0^y [1 - \frac{ax^2}{y^2}] \left[x^3 [J_0(x) - J_2(x)] Cos[(\frac{Dx}{2y})^4 \frac{k}{8R}(R^{-2} - F^{-2})] \right. \\ \left. - \frac{kD^2x^2}{2R} \left(\frac{x}{y} \right)^2 J_1(x) Sin[(Dx/2y)^4 \frac{k}{8R}(R^{-2} - F^{-2})] \right] xdx J_0 \left(\frac{2ry}{D} \right) dy$$

(27)

We have found no convenient analytic expression for this double integral and we have evaluated it numerically.

NUMERICAL SIMULATIONS

When, as often in practice, the data window is square, the expressions are far from tractable and we have preferred to use numerical simulations to estimate numerical values.

A model aperture with Gaussian illumination and constant phase was used. The simulations were based on the idea of calculating the exact radiated field on an array of points in the near field. Then the aperture field was calculated by Fast Fourier Transform and simple Fresnel correction. Comparison of this approximate solution with the original model then gave the errors in the simulated holographic process due to neglect of the higher order terms in the Fresnel correction (equation 4). The effects of small data windows were simulated by masking the outer parts of the calculated near field to zero.

For small data windows, comparable to the geometric optics beam (K =1), the aperture phase errors are dominated by the effects of the resulting truncation of the data. The largest errors are always concentrated towards the edge of the aperture with a distribution reflecting the shape of the truncation. At larger values of K this error distribution retreats to the rim and becomes a circular "ringing" response. This was reported by McGrane (1983). At sufficiently large values of K the effects of the errors in the Fresnel correction become apparent as they come to dominate the ringing phenomena, now smaller. For the case of a square data window they have a fourfold symmetry, reflecting the shape of the data window, with again maxima at the rim.

REFERENCES

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Silver S.,"Microwave Antenna Theory and Design",Peter Perigrins,London,1984
Southwell W.H., J.Opt.Soc.Am. 71,7,1981

APPENDIX

TABLE 1 Surface error estimates for ALMA antennas (microns)

(For a telescope of diameter 12m, with -14 dB taper, measured at 3.25mm wavelength at a distance of 300m from the transmitter)

	N=128			N=256			Source
Infocus							
	Peak	RMS	WRMS	Peak	RMS	WRMS	
Square Window	28			85			equ 10
Circular Window	16			44			equ 10
Ideal Refocus							
Circular Window	7	1.2	1	28	5	4	equ 21-23
Infocus							
Square Window	3.3	0.5	0.4				simulation
Circular Window							simulation
Refocus							
Square Window	2.6	1.0	0.9				simulation
Circular Window							simulation