Appendix F

The Radiometer Equation for Position Switched Measurements

Defining the following

$$
\begin{aligned}
T_{rms}^{pos} &\equiv \text{rms noise level for an } \mathsf{ON}, \mathsf{OFF} \text{ or } \mathsf{GAIN} \text{ scan measurment} \\
&\equiv \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu t_{pos}}} \qquad \text{(F.1)} \\
\mathsf{SCAN} &\equiv \frac{\mathsf{ON}-\mathsf{OFF}}{\mathsf{OFF}} \times \mathsf{GAIN} \qquad \text{(F.2)} \\
\alpha &\equiv \frac{t_{off}}{t_{on}} \qquad \text{(F.3)} \\
\eta_{spec} &\equiv \text{spectrometer efficiency (see Table F.1)} \qquad \text{(F.4)} \\
& \qquad \text{(F.5)}
\end{aligned}
$$

we can calculate $(T_{rms}^{scan})^2$

$$
\begin{aligned}
(T_{rms}^{scan})^2 &= \left(\frac{\mathsf{ON}-\mathsf{OFF}}{\mathsf{OFF}} \times \mathsf{GAIN}\right)^2 \left\{\frac{(T_{rms}^{on})^2 + (T_{rms}^{off})^2}{\left(\mathsf{ON}-\mathsf{OFF}\right)^2} + \frac{(T_{rms}^{off})^2}{\mathsf{OFF}^2} + \frac{(T_{rms}^{gain})^2}{\mathsf{GAIN}^2}\right\} \\
&= \left(\frac{\mathsf{GAIN}}{\mathsf{OFF}}\right)^2 \left((T_{rms}^{on})^2 + (T_{rms}^{off})^2\right) \qquad \text{(F.6)}
\end{aligned}
$$

where I have neglected terms quadratic in $\frac{\mathsf{ON}-\mathsf{OFF}}{\mathsf{OFF}}$. Noting that

$$(T_{rms}^{on-off})^2 = (T_{rms}^{on})^2 + (T_{rms}^{off})^2 \qquad \text{(F.7)}$$

we see that Equation F.7 and Equation F.6 differ by only a constant (which is the volt-to-temperature scaling factor). Therefore

$$\begin{aligned} T_{rms}^{scan} &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu}}\left[\frac{1}{t_{on}} + \frac{1}{t_{off}}\right]^{\frac{1}{2}} \\ &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu}}\left[\frac{1}{t_{on}} + \frac{1}{\alpha t_{on}}\right]^{\frac{1}{2}} \\ &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu t_{on}}}\left[1 + \frac{1}{\alpha}\right]^{\frac{1}{2}} \end{aligned} \tag{F.8}$$

If one is taking multiple ON scans per OFF scan, then the time spent to acquire a scan is given by

$$\begin{aligned} t_{scan} &= t_{on} + \frac{t_{off}}{N} \\ &= \left(1 + \frac{\alpha}{N}\right)t_{on} \end{aligned} \tag{F.9}$$

where N is the number of ON source measurements per OFF source measurement. Therefore, T_{rms}^{scan} becomes

$$\begin{aligned} T_{rms}^{scan} &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu\left(\frac{t_{scan}}{1+\frac{\alpha}{N}}\right)}}\left(1 + \frac{1}{\alpha}\right)^{\frac{1}{2}} \\ &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu t_{scan}}}\left[\left(1 + \frac{\alpha}{N}\right)\left(1 + \frac{1}{\alpha}\right)\right]^{\frac{1}{2}} \end{aligned} \tag{F.10}$$

To determine the optimum OFF source integration time time when one is acquiring multiple ON source measurements per OFF source measurement, we find the minimum of Equation F.10 with respect to α:

$$\begin{aligned} \frac{dT_{rms}^{scan}}{d\alpha} &= \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu t_{scan}}}\left\{\frac{1}{2N}\left(1 + \frac{\alpha}{N}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{\alpha}\right)^{\frac{1}{2}} - \frac{1}{2\alpha^2}\left(1 + \frac{\alpha}{N}\right)^{\frac{1}{2}}\left(1 + \frac{1}{\alpha}\right)^{-\frac{1}{2}}\right\} \\ &= 0 \end{aligned} \tag{F.11}$$

which reduces to

$$\begin{aligned} \left(\frac{1}{N}\right)\left(\frac{N}{N+\alpha}\right)^{\frac{1}{2}}\left(1 + \frac{1}{\alpha}\right)^{\frac{1}{2}} &= \left(\frac{1}{\alpha^2}\right)\left(\frac{N+\alpha}{N}\right)^{\frac{1}{2}}\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} \\ \alpha^2 &= N \\ t_{off}^{optimal} &= \sqrt{N}t_{on} \end{aligned} \tag{F.12}$$

Table F.1: 2-Bit Correlator Efficiencies

Number of Levels	η_{spec}[1]	TMS Equation and Page
2	$\frac{2}{\pi}$	8.31, page 220
3	0.809	8.58, page 228
4	0.881	8.50, page 224
5	0.920	...
8	0.962	...

where I have used Equation F.3. Inserting this value for $t_{off}^{optimal}$ into Equation F.10 leads to the following relation for the optimal scan rms when one observes N ON source measurements per OFF source measurement

$$^{optimal}T_{rms}^{scan} = \frac{T_{sys}}{\eta_{spec}\sqrt{\Delta\nu t_{scan}}}\left(1 + \frac{1}{\sqrt{N}}\right) \tag{F.13}$$

See Ball (1976, "Methods of Experimental Physics", volume 12, part C, page 52) for a parallel discussion on noise considerations for position switched measurements.

The factor η_{spec} takes account of the efficiency for quantized correlation spectrometers like the 12m MAC. Cooper (1976, "Methods of Experimental Physics", volume 12, part B, page 284) discusses this correlator efficiency and quotes the η_{spec} values noted in Table F.1.

Note that the quoted values assume Nyquist sampling. Many correlators, including the MAC, oversample by some amount, so the correlator efficiency is actually somewhat higher than that for Nyquist sampled data. The theoretical calculations are always predicated on having rectangular passbands of exactly the nominal bandwidth, whereas the effective bandwidth with practical filters is always smaller. This and the fact that there are various non-ideal properties of digitizers will tend to reduce the efficiency. Thus, for the purpose of estimating the required observing time to reach a given sensitivity, users should use the theoretical efficiency for Nyquist sampling since the benefit of oversampling will be cancelled by miscellaneous other losses. Formulas for calculating the correlator efficiency for arbitrary oversampling factors can be found in Thompson, Moran, and Swenson (1986) and are listed in Table F.1. For the MAC, which is a 2-bit, 3-level correlator, the correlator efficiency is plotted in Figure F.1.

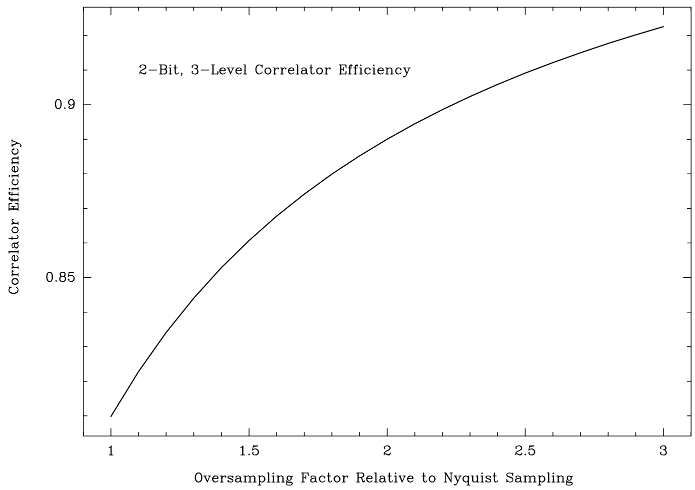

Figure F.1: Correlator efficiency for the MAC (a 2-bit, 3-level correlator) as a function of oversampling. Note that for the MAC, which is oversampled by a factor of $\frac{4}{3}$ over Nyquist, the efficiency is 0.847.