Collected Transmission-Line Identities

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1 Introduction

Transmission-line identities are a useful tool in the early stages of passive network design, and many such identities are well known and reported in the literature. To the author's knowledge, however, there is no single source a researcher can go to for a comprehensive list of all known identities, including those less commonly used, for easy reference. What follows is an attempt to collect all known transmission-line identities into one place to serve that purpose.

This catalog is organized into identity sets—groups of two or more networks that are all equivalent to one another with the proper selection of element values. Schematics of the equivalent networks are shown in the first row, with the formulas for calculating the element values in the space below. All transmission-lines are assumed to have equal electrical length (θ), unless otherwise noted (e.g. 2θ or $\theta/2$). Furthermore, in all cases, ρ shall represent the ratio of even-mode to odd-mode impedance (e.g., $\rho_2 = z_{e2}/z_{o2} = y_{o2}/y_{e2}$, etc.) Both impedance and admittance expressions have been used where appropriate to keep the formulas as compact as possible, and it shall be understood that impedance and admittance parameters having common subscripts are inverses of one another (that is, $y_{e2} = 1/z_{e2}$). Two-port identity sets are covered in Section 2, three-port sets in Section 3, and four-ports in Section 4.

The intent is for this to be a living document, and to make it as complete as possible. To that end, the author welcomes feedback regarding any identities that may be missing, corrections to formulas, or references for identities that are already included if none is provided or an earlier one is found. Such contributions will be gratefully acknowledged and cited.

A few notes about what constitutes an "identity": An identity is a pair of circuits or networks for which all port parameters are mathematically identical at all frequencies. Approximate matches are not considered identities, nor are networks which match only at discrete frequencies. For brevity, this document will not include trivial combinations or repetition of identities unless they are particularly revealing. Nor will it include networks which depend on hypothetical elements with no practical realization, such as all-frequency quadrature hybrids, ideal dispersionless impedance/admittance inverters, or negative-length transmission lines, unless the identity enables the elimination of such elements to make the networks realizable.

2 Two-Port Identities

(a) (b) (c) (d)

$$z_1 = z_a\left(1 + \frac{z_a}{z_b}\right) = \frac{1}{2} z_{e1}\left(1 + \frac{z_{e1}}{z_{o1}}\right) = 2 z_{e2} z_{o2} \frac{z_{e2}+z_{o2}}{(z_{e2}-z_{o2})^2}$$

$$z_2 = z_a + z_b = \frac{1}{2}\left(z_{e1} + z_{o1}\right) = \frac{1}{2}\left(z_{e2} + z_{o2}\right)$$

$$z_a = \frac{z_1 z_2}{z_1+z_2} = \frac{1}{2} z_{e1} = \frac{2 z_{e2} z_{o2}}{z_{e2}+z_{o2}} \qquad z_b = \frac{z_2^2}{z_1+z_2} = \frac{1}{2} z_{o1} = \frac{1}{2}\frac{(z_{e2}-z_{o2})^2}{z_{e2}+z_{o2}}$$

$$z_{e1} = 2\frac{z_1 z_2}{z_1+z_2} = 2 z_a = \frac{4 z_{e2} z_{o2}}{z_{e2}+z_{o2}} \qquad z_{o1} = 2\frac{z_2^2}{z_1+z_2} = 2 z_b = \frac{(z_{e2}-z_{o2})^2}{z_{e2}+z_{o2}}$$

$$z_{e2} = z_2\left(1 + \sqrt{\frac{z_2}{z_1+z_2}}\right) = z_a + z_b + \sqrt{z_b\left(z_a + z_b\right)}$$

$$= \frac{1}{2}\left(z_{e1} + z_{o1}\right) + \frac{1}{2}\sqrt{z_{o1}\left(z_{e1} + z_{o1}\right)}$$

$$z_{o2} = z_2\left(1 - \sqrt{\frac{z_2}{z_1+z_2}}\right) = z_a + z_b - \sqrt{z_b\left(z_a + z_b\right)}$$

$$= \frac{1}{2}\left(z_{e1} + z_{o1}\right) - \frac{1}{2}\sqrt{z_{o1}\left(z_{e1} + z_{o1}\right)}$$

Source: (a)=(b)=(d) [1–3], (a,b,d)=(c) [4, 5]

(a) (b) (c) (d)

$$z_1 = 2\frac{z_{e1} z_{o1}}{z_{e1}+z_{o1}} = \frac{1}{2}\left(z_{e2} + z_{o2}\right) = \frac{1}{2} z_{e3}$$

$$z_{e1} = \frac{1}{2} z_1\left(1 + \rho_1\right) = \frac{1}{4}\left(z_{e2} + z_{o2}\right)\left(1 + \rho_1\right) = \frac{1}{4} z_{e3}\left(1 + \rho_1\right)$$

$$z_{o1} = \frac{1}{2} z_1\left(1 + \rho_1^{-1}\right) = \frac{1}{4}\left(z_{e2} + z_{o2}\right)\left(1 + \rho_1^{-1}\right) = \frac{1}{4} z_{e3}\left(1 + \rho_1^{-1}\right)$$

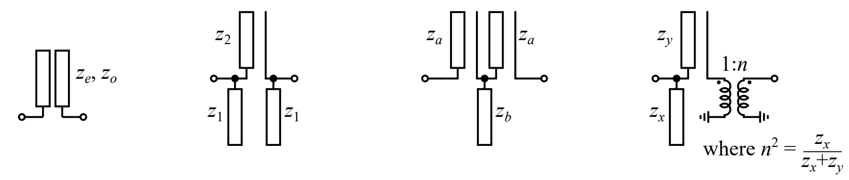
for any ρ_1

$$z_{e2} = 2 z_1 \frac{\rho_2}{1+\rho_2} = 4\frac{z_{e1} z_{o1}}{z_{e1}+z_{o1}}\frac{\rho_2}{1+\rho_2} = z_{e3}\frac{\rho_2}{1+\rho_2}$$

$$z_{o2} = 2 z_1 \frac{1}{1+\rho_2} = 4\frac{z_{e1} z_{o1}}{z_{e1}+z_{o1}}\frac{1}{1+\rho_2} = z_{e3}\frac{1}{1+\rho_2} \qquad \text{for any } \rho_2$$

$$z_{e3} = 2 z_1 = 4\frac{z_{e1} z_{o1}}{z_{e1}+z_{o1}} = z_{e2} + z_{o2} \qquad \text{for any } z_{o3}$$

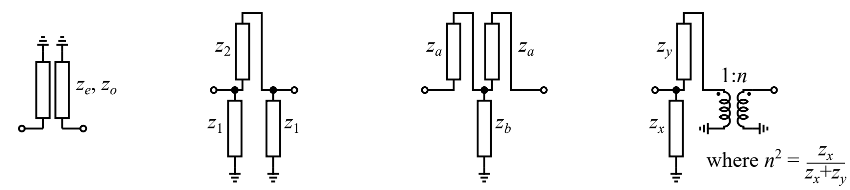
Source: (a)=(b)=(c) [1–3], (a-c)=(d) [4, 5]



(a) (b) (c) (d)

$z_e = z_1 = z_a + 2z_b = z_x(n+1)$	$z_o = \frac{z_1 z_2}{2z_1+z_2} = z_a = z_y \frac{n^2}{n+1}$
$z_1 = z_e = z_a + 2z_b = z_x(n+1)$	$z_2 = \frac{2z_e z_o}{z_e-z_o} = (z_a + 2z_b)\,\frac{z_a}{z_b} = z_y n$
$z_a = z_o = \frac{z_1 z_2}{2z_1+z_2} = z_y \frac{n^2}{n+1}$	$z_b = \frac{1}{2}\,(z_e - z_o) = \frac{z_1^2}{2z_1+z_2} = z_x n$
$z_x = \frac{1}{2}\,(z_e + z_o) = \frac{z_1}{n+1} = \frac{z_b}{n}$	$z_y = 2z_e z_o \frac{z_e+z_o}{(z_e-z_o)^2} = \frac{z_2}{n} = z_a \frac{n+1}{n^2}$
$n = \frac{z_e-z_o}{z_e+z_o} = \frac{z_1}{z_1+z_2} = \frac{z_b}{z_a+z_b}$	

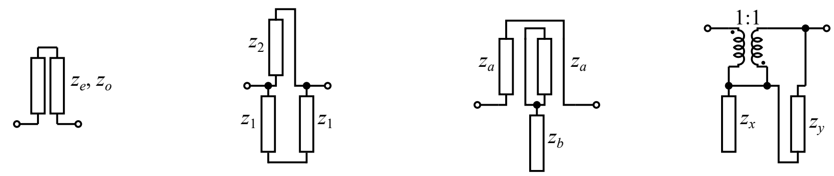
Source: (a)=(c) [1–3], (a,c)=(b)=(d) [4, 5]



(a) (b) (c) (d)

$z_e = z_1 = z_a + 2z_b = z_x(n+1)$	$z_o = \frac{z_1 z_2}{2z_1+z_2} = z_a = z_y \frac{n^2}{n+1}$
$z_1 = z_e = z_a + 2z_b = z_x(n+1)$	$z_2 = \frac{2z_e z_o}{z_e-z_o} = (z_a + 2z_b)\,\frac{z_a}{z_b} = z_y n$
$z_a = z_o = \frac{z_1 z_2}{2z_1+z_2} = z_y \frac{n^2}{n+1}$	$z_b = \frac{1}{2}\,(z_e - z_o) = \frac{z_1^2}{2z_1+z_2} = z_x n$
$z_x = \frac{1}{2}\,(z_e + z_o) = \frac{z_1}{n+1} = \frac{z_b}{n}$	$z_y = 2z_e z_o \frac{z_e+z_o}{(z_e-z_o)^2} = \frac{z_2}{n} = z_a \frac{n+1}{n^2}$
$n = \frac{z_e-z_o}{z_e+z_o} = \frac{z_1}{z_1+z_2} = \frac{z_b}{z_a+z_b}$	

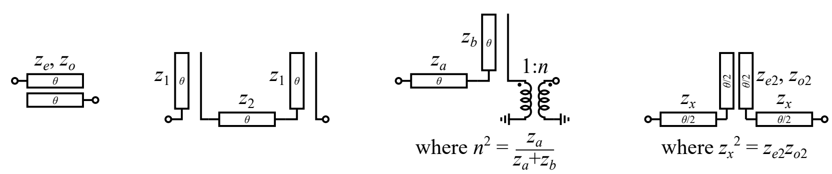
Source: (a)=(b) [1–3], (a,b)=(c) [3], (a-c)=(d) [4, 5]



(a) (b) (c) (d)

$z_e = z_1 = z_a + 2z_b = 2z_x$	$z_o = \frac{z_1 z_2}{2z_1+z_2} = z_a = 2z_y$
$z_1 = z_e = z_a + 2z_b = 2z_x$	$z_2 = 2\frac{z_e z_o}{z_e - z_o} = \frac{z_a}{z_b}\left(z_a + 2z_b\right) = 4\frac{z_x z_y}{z_x - z_y}$
$z_a = z_o = \frac{z_1 z_2}{2z_1+z_2} = 2z_y$	$z_b = \frac{1}{2}\left(z_e - z_o\right) = \frac{z_1^2}{2z_1+z_2} = z_x - z_y$
$z_x = \frac{1}{2}z_e = \frac{1}{2}z_1 = \frac{1}{2}\left(z_a + 2z_b\right)$	$z_y = \frac{1}{2}z_o = \frac{1}{2}\frac{z_1 z_2}{2z_1+z_2} = \frac{1}{2}z_a$

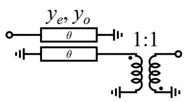
Source: (a)=(d) [1,2], (a,d)=(b)=(c) [4,5]



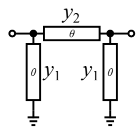
(a) (b) (c) (d)

$z_e = z_1 + 2z_2 = z_a(1+n) = 2z_x\frac{z_{e2}-z_x}{z_{e2}-z_{o2}}$ $z_o = z_1 = z_a(1-n) = 2z_x\frac{z_x - z_{o2}}{z_{e2}-z_{o2}}$
$z_1 = z_o = z_a(1-n) = 2z_x\frac{z_x - z_{o2}}{z_{e2}-z_{o2}}$ $z_2 = \frac{1}{2}\left(z_e - z_o\right) = z_a n = z_x\frac{z_{e2}+z_{o2}-2z_x}{z_{e2}-z_{o2}}$
$z_a = \frac{1}{2}\left(z_e + z_o\right) = z_1 + z_2 = z_x$ $z_b = 2z_e z_o\frac{z_e+z_o}{(z_e-z_o)^2} = z_1\left(1+\frac{z_1}{z_2}\right)\left(2+\frac{z_1}{z_2}\right) = 4z_x\frac{(z_x-z_{o2})(z_{e2}-z_x)}{(z_{e2}+z_{o2}-2z_x)^2}$ $n = \frac{z_e - z_o}{z_e+z_o} = \frac{z_2}{z_1+z_2} = \frac{z_{e2}+z_{o2}-2z_x}{z_{e2}-z_{o2}}$
$z_x = \frac{1}{2}\left(z_e + z_o\right) = z_1 + z_2 = z_a$ $z_{e2} = \frac{1}{2}\left(z_e + z_o\right)\frac{z_e}{z_o} = \left(z_1 + z_2\right)\frac{z_1+2z_2}{z_1} = z_a\frac{1+n}{1-n}$ $z_{o2} = \frac{1}{2}\left(z_e + z_o\right)\frac{z_o}{z_e} = \left(z_1 + z_2\right)\frac{z_1}{z_1+2z_2} = z_a\frac{1-n}{1+n}$

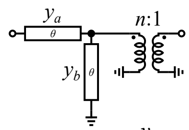
Source: (a)=(b) [1–3], (a,b)=(d) [6], (a,b,d)=(c) [4,5]



(a)

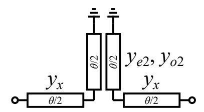


(b)



where $n^2 = \frac{y_a}{y_a + y_b}$

(c)



where $y_x{}^2 = y_{e2} y_{o2}$

(d)

$$y_e = y_1 = y_a\left(1-n\right) = 2y_x \tfrac{y_x - y_{e2}}{y_{o2}-y_{e2}}$$

$$y_o = y_1 + 2y_2 = y_a\left(1+n\right) = 2y_x \tfrac{y_{o2} - y_x}{y_{o2}-y_{e2}}$$

$$y_1 = y_e = y_a\left(1-n\right) = 2y_x \tfrac{y_x - y_{e2}}{y_{o2}-y_{e2}}$$

$$y_2 = \tfrac{1}{2}\left(y_o - y_e\right) = y_a n = y_x \tfrac{y_{o2}+y_{e2}-2y_x}{y_{o2}-y_{e2}}$$

$$y_a = \tfrac{1}{2}\left(y_o + y_e\right) = y_1 + y_2 = y_x$$

$$y_b = 2y_e y_o \tfrac{y_o + y_e}{\left(y_o - y_e\right)^2} = y_1\left(1 + \tfrac{y_1}{y_2}\right)\left(2 + \tfrac{y_1}{y_2}\right) = 4y_x \tfrac{\left(y_x - y_{e2}\right)\left(y_{o2}-y_x\right)}{\left(y_{o2}+y_{e2}-2y_x\right)^2}$$

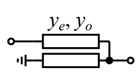
$$n = \tfrac{y_o - y_e}{y_o + y_e} = \tfrac{y_2}{y_1 + y_2} = \tfrac{y_{o2}+y_{e2}-2y_x}{y_{o2}-y_{e2}}$$

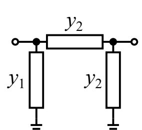
$$y_x = \tfrac{1}{2}\left(y_e + y_o\right) = y_1 + y_2 = y_a$$

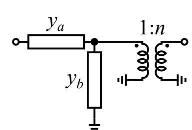
$$y_{e2} = \tfrac{1}{2}\left(y_e + y_o\right)\tfrac{y_e}{y_o} = \left(y_1 + y_2\right)\tfrac{y_1}{y_1 + 2y_2} = y_a \tfrac{1-n}{1+n}$$

$$y_{o2} = \tfrac{1}{2}\left(y_e + y_o\right)\tfrac{y_o}{y_e} = \left(y_1 + y_2\right)\tfrac{y_1 + 2y_2}{y_1} = y_a \tfrac{1+n}{1-n}$$

Source: (a)=(b) [1–3], (a,b)=(d) [7], (a,b,d)=(c) [4, 5]







where $2n = 1 + \frac{y_b}{y_a}$

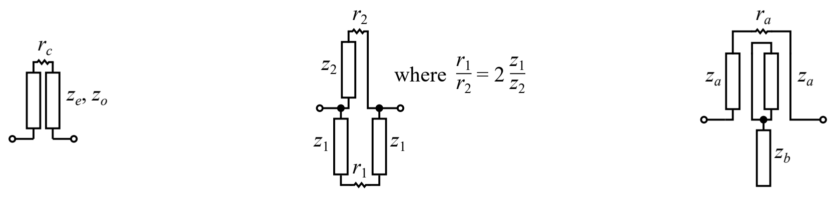
$$y_e = y_2 = y_a n^{-1} \qquad y_o = 2y_1 + y_2 = y_b n^{-1}$$

$$y_1 = \tfrac{1}{2}\left(y_o - y_e\right) = \tfrac{1}{2}\left(y_b - y_a\right)n^{-1} \qquad y_2 = y_e = y_a n^{-1}$$

$$y_a = \tfrac{1}{2} y_e\left(1+\rho\right) = y_1 + y_2$$

$$y_b = \tfrac{1}{2} y_o\left(1+\rho\right) = \left(y_1 + y_2\right)\left(1 + 2\tfrac{y_1}{y_2}\right) \qquad n = \tfrac{1}{2}(1+\rho) = 1 + \tfrac{y_1}{y_2}$$

Source: [4, 5]



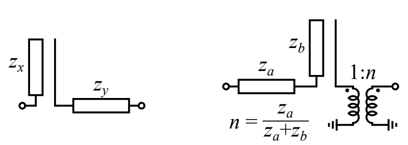
$z_e = z_1 = z_a + 2z_b \qquad z_o = \frac{z_1 z_2}{2z_1+z_2} = z_a \qquad r_c = \frac{r_1 r_2}{r_1+r_2} = r_a$

$z_1 = z_e = z_a + 2z_b \qquad z_2 = 2\frac{z_e z_o}{z_e - z_o} = \frac{z_a}{z_b}\left(z_a + 2z_b\right)$

$r_1 = \frac{z_e}{z_o} r_c = \left(1 + 2\frac{z_b}{z_a}\right) r_a \qquad r_2 = \frac{z_e}{z_e - z_o} r_c = \left(1 + \frac{z_a}{2z_b}\right) r_a$

$z_a = z_o = \frac{z_1 z_2}{2z_1+z_2} \qquad z_b = \frac{1}{2}\left(z_e - z_o\right) = \frac{z_1}{z_2}\frac{z_1 z_2}{2z_1+z_2} \qquad r_a = r_c = \frac{r_1 r_2}{r_1+r_2}$

Source: [4, 5]

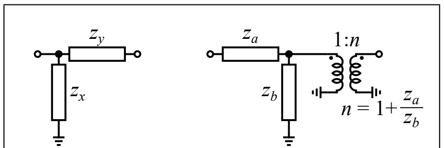


$z_x = z_b n \qquad z_y = z_a n$

$z_a = z_x + z_y \qquad z_b = \left(z_x + z_y\right)\frac{z_x}{z_y}$

$n = \frac{z_y}{z_x+z_y}$

Source: [1–3]

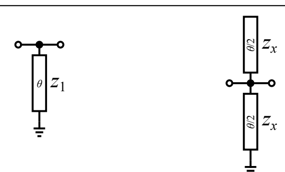


$z_x = z_b n \qquad z_y = z_a n$

$z_a = \frac{z_x z_y}{z_x+z_y} \qquad z_b = \frac{z_x^2}{z_x+z_y}$

$n = 1 + \frac{z_y}{z_x}$

Source: [1–3]

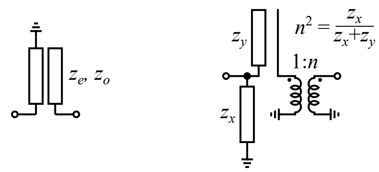


$z_1 = \frac{1}{2}z_x$	$z_x = 2z_1$

Source: [8]

$z_1 = 2z_x$	$z_x = \frac{1}{2}z_1$

Source: [8]

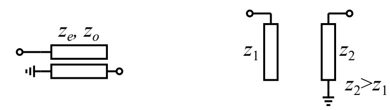


$$z_e = z_x(n+1) \qquad z_o = z_y \tfrac{n^2}{n+1}$$

$$z_x = \tfrac{1}{2}\left(z_e + z_o\right)$$

$$z_y = 2 z_e z_o \tfrac{z_e+z_o}{(z_e-z_o)^2} \qquad n = \tfrac{z_e-z_o}{z_e+z_o}$$

Source: [1–3]

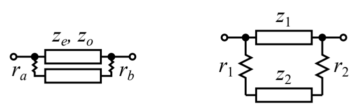


$$z_e = z_2 + \sqrt{z_2\left(z_2 - z_1\right)}$$

$$z_o = z_2 - \sqrt{z_2\left(z_2 - z_1\right)}$$

$$z_1 = 2\tfrac{z_e z_o}{z_e+z_o} \qquad z_2 = \tfrac{1}{2}\left(z_e + z_o\right)$$

Source: [1–3]



$$z_e = 2\tfrac{z_1 z_2}{z_1+z_2} \qquad r_a = 4 r_1 \left(\tfrac{z_1}{z_1+z_2}\right)^2$$

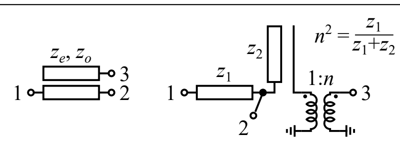
$$z_o = 2\tfrac{z_1^2}{z_1+z_2} \qquad r_b = 4 r_2 \left(\tfrac{z_1}{z_1+z_2}\right)^2$$

$$z_1 = \tfrac{1}{2}\left(z_e + z_o\right) \qquad r_1 = \tfrac{r_a}{4}\left(\rho+1\right)^2$$

$$z_2 = \tfrac{1}{2}\rho\left(z_e + z_o\right) \qquad r_2 = \tfrac{r_b}{4}\left(\rho+1\right)^2$$

Source: [4, 5]

3 Three-Port Identities

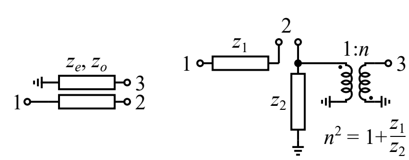


$$z_e = z_1(1+n) \qquad z_o = z_1(1-n)$$

$$z_1 = \tfrac{1}{2}\left(z_e + z_o\right)$$

$$z_2 = 2 z_e z_o \tfrac{z_e+z_o}{(z_e-z_o)^2} \qquad n = \tfrac{z_e-z_o}{z_e+z_o}$$

Source: [9]

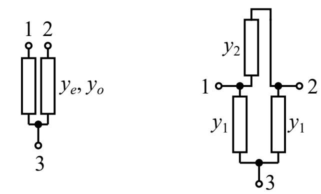


$$z_e = z_2 n(n+1) \qquad z_o = z_2 n(n-1)$$

$$z_1 = 2\tfrac{z_e z_o}{z_e+z_o}$$

$$z_2 = \tfrac{1}{2}\tfrac{(z_e-z_o)^2}{z_e+z_o} \qquad n = \tfrac{z_e+z_o}{z_e-z_o}$$

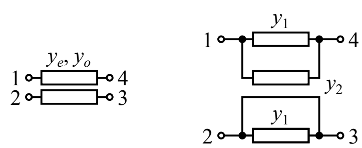
Source: [4, 5]



$y_e = y_1$	$y_o = y_1 + 2y_2$
$y_1 = y_e$	$y_2 = \frac{1}{2}(y_o - y_e)$

Source: [4, 5]

4 Four-Port Identities



$y_e = y_1$	$y_o = y_1 + 2y_2$
$y_1 = y_e$	$y_2 = \frac{1}{2}(y_o - y_e)$

Source: [4, 5]

5 Example Use Cases

The useful application of the identities listed in this document are made more clear by a few schematic examples.

5.1 Kuroda's Identity

One of the most common uses of a transmission-line identity is to eliminate series-connected stubs for microstrip or stripline circuits, where they are difficult to realize in practice. In the example of Figure 1, the famous Kuroda's Identity is used to replace a short-circuited, series stub with an open-circuited parallel stub [1–3,5]. This particular case arises as a result of Richard's Transformation applied to a low-pass, lumped-element prototype. Multiple applications of Kuroda's identity are usually required here; only the last of several such applications needed in this circuit are shown for brevity.

5.2 Inverter Polarity

In another example, shown in Figure 2, two different identities for Π-type line-and-stub subnetworks are used to realized couplings with opposite sign in a

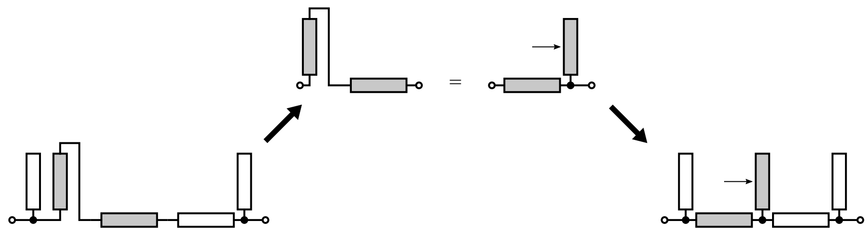


Figure 1: Example application of Kuroda's identity to eliminate series-connected stubs.

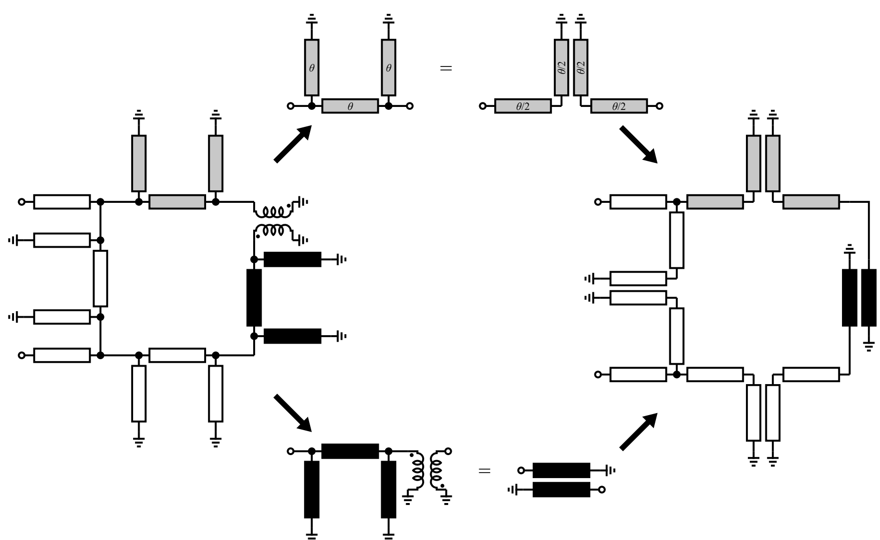


Figure 2: Two different identities are applied to Π-type line-and-stub subnetworks to achieve couplings with opposite signs [7].

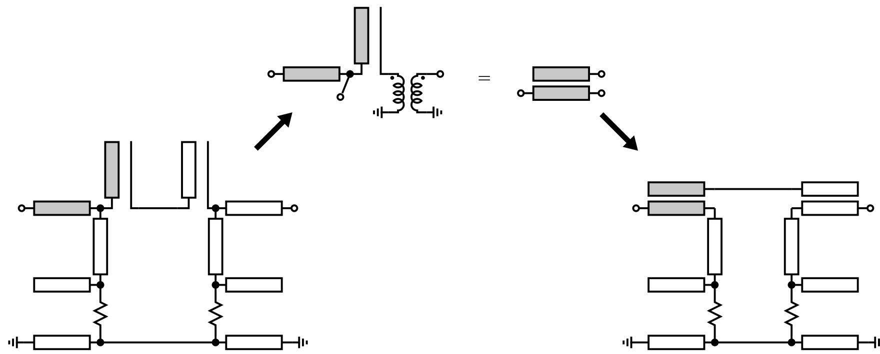


Figure 3: A three-port identity is used to eliminate series-connected stubs between two branch-points in a reflectionless filter.

cross-coupled resonator filter, as well as to reduce crowding of multiple parallel stubs [7]. Several instances of the first identity are applied in this example.

5.3 Reflectionless Filter

Finally, the reflectionless filter in Figure 3 is another example where series stubs had to be removed in order to arrive at a realizable circuit. The position of those stubs in this case, trapped between two branch points, required the use of a three-port identity [9]. The temporary insertion of back-to-back transformers before both are eliminated is implicit in the application of this identity.

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